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A Completeness of Metrics for Topological Relations in 3D Qualitative Spatial Reasoning

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***Abstract*—For qualitative spatial reasoning, there are various dimensions of objects. A considerable amount of effort has been devoted to 2D representation and analysis of spatial relations. Here we present an exposition for 3D objects. There are three types of binary relations between pairs of objects: topological connectivity, cardinal directions, and distance relations. The combinations of these relations can provide additional useful knowledge. The spatial databases include data and the spatial relations to facilitate end-user spatial querying, it also is important to associate natural language with these relations. Some work has been done in this regard for line-region and region-region topological relations in 2D, and very recent work has initiated the association between natural language, topology, and metrics for 3D objects. However, prior efforts have lacked rigorous analysis, expressive power, and completeness of the associated metrics. Herein we present a detailed study of new metrics required to bridge the gap between topological connectivity and size information for integrating reasoning in spatial databases. The complete set of metrics that we present should be useful for a variety of applications dealing with 3D objects including regions with vague boundaries.**

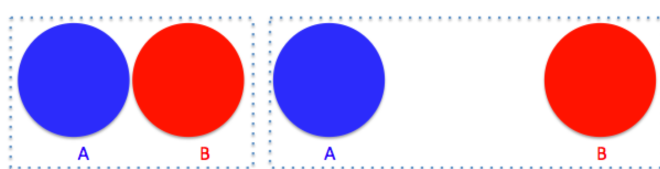
***Keywords*—Region connection calculus, metrics, spatial reasoning, qualitative reasoning.**

I. Introduction

SPATIAL relation theory is the foundation for using spatial databases effectively. In qualitative spatial reasoning, there are various types of object dimensions. A considerable amount of effort has been devoted to 2D. Here we present exposition for 3D objects. There are three types of binary relations between pairs of objects. The combinations of these relations can provide additional useful knowledge required for some applications. The spatial databases include data and the spatial relations to facilitate end-user spatial querying and retrieving and analyzing spatial knowledge quickly. Qualitative spatial reasoning is intrinsically useful even when spatial information is imprecise or incomplete. The reasons are: (1) precise information may not be available or may not even be required, (2) detailed parameters may not be necessary before proceeding to decision making, and (3) complex decisions sometimes must be made in a relatively short period of time.

However, qualitative reasoning can result in ambiguous solutions due to incomplete or imprecise quantitative information. In RCC8 [1], [2], the regions have a well-defined interior, boundary, and exterior. The RCC8 relations are bivalent with true and false crisp values. Mathematically defined and computer drawn objects are crisp and well defined, whereas hand-drawn regions tend to have a vague boundary [3]. When regions are vague, the relations between regions can be vague also. That results in the possible values for relations being true, false, or even 'maybe.' RCC8 assumes that regions are crisp; hence the relations are crisp. While topology is sufficient to determine the spatial connectivity relations, it lacks the capability to determine the degree (or extent) of connectivity or separation of such relations.

For example, in Fig. 1, for two objects A and B, the RCC8 disjoint relation, DC(A,B), evaluates to true, yet it does not provide any information about the degree of separation; we do not know how close the two objects are — are they almost touching or are they far apart? The usefulness of metrics lies in providing such additional information, which can be useful in some applications.



(a) (b)

Fig. 1. Two examples of disjoint objects. RCC8 determines that A and B are disjoint, yet it does not tell if they are almost touching or far apart. That is, it does not quantify the degree of disjointness.

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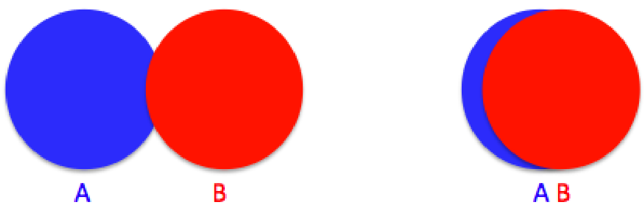
For another example, in Fig. 2, for two objects A and B, the RCC8 proper overlap relation, PO(A,B), evaluates to true, yet it does not provide any information about the degree of connectivity; we do not know how much is the overlap—are the objects barely overlapping or are they are almost equal?



(a) (b)

Fig. 2. Two examples of proper overlap. RCC8 determines that there is an overlap between A and B, but it does not quantify the proper overlap. In (a) the objects are barely overlapping, and in (b) they are almost equal.

Similarly in Fig. 2, a metric can determine which object is larger or if they are equal independent of location. Metrics are quantitative, whereas topology is qualitative; both together can supplement each other in terms of spatial knowledge. The metric refinements provide for quality of connectivity of each relation. The goal of this exposition is to bridge the gap between topology and size via metrics.

The paper is organized as follows. Section II provides a brief mathematical background relevant to subsequent discussions in the paper. Section III explains the motivation for metrics. Section IV discusses the development of our metrics, as well as the association between size and topology. Section V explains the association between connectivity, size, and metrics. Section VI gives the conclusion and future directions.

II. Background

A. Spatial Relations in General

Historically, there are two approaches to topological region connection calculus, one is based on first order logic [1], and the second is based on the 9-intersection model [2]. Both of these approaches assume that regions are in 2D and the regions are crisp, and that relation membership values are true and false only. For qualitative distances, metrics were used in 1D to differentiate relative terms of proximity like *very close*, *close*, *far*, and *very far* [4]. To refine natural language ontology and topological relationships, metrics were introduced for line-region and region-region connectivity in 2D [5]. These approaches lack expressing the strength of relation, and the combination of the connectivity and size information. Recently more attention has been directed to these issues for vague regions in 2D [6] and for natural language ontology in 3D [7]. However, prior work has been deficient in rigorous analysis, expressive power, and completeness of the metrics. The complete set of metrics presented herein differs from the previous approaches in its completeness and enhanced expressiveness.

B. Mathematical Preliminaries

R^3 denotes the three-dimensional space endowed with a distance metric. Here the mathematical notions of *subset, proper subset, equal sets, empty set* (∅), *union, intersection, universal complement*, and *relative complement* are the same as those typically defined in set theory. The notions of *neighborhood*, *open set*, *closed set*, *limit point*, *boundary*, *interior*, *exterior*, and *closure* of sets are as in point-set topology. The interior, boundary, and exterior of any region are disjoint, and their union is the universe.

A set is *connected* if it cannot be represented as the union of disjoint open sets. For any non-empty bounded set A, we use symbols A^c, A^i, A^b, and A^e to represent the universal complement, interior (Int(A)), boundary (Bnd(A)), and exterior (Ext(A)) of a set A, respectively. Two regions A and B are equal if $A^i == B^i$, $A^b == B^b$, and $A^e == B^e$ are true. For our discussion, we assume that every region A is a non-empty, bounded, regular closed, connected set without holes; specifically, A^b is a closed curve in 2D, and a closed surface in 3D. A spatial region A is closed if it contains the interior and boundary, and is denoted by $\overline{A}$. Thus the regions are regular closed sets, meaning closure of the interior of a region is itself, $A = \overline{A^i}$. For spatial regions, we use *weak* connectivity: two regions A and B are connected if $\overline{A} \cap B \neq \emptyset$.

C. Metric Spaces and Spatial Metrics

Topologically a metric m on a metric space satisfies three properties: (1) $m(A, A) = 0$, *identity*; (2) $m(A, B) = m(B, A)$, *symmetry*; and (3) $m(A, B) \leq m(A, C) + m(C, B)$, *triangle inequality*. Furthermore a metric is translation invariant if $m(A,B) = m(A + t, B + t)$ where A + t is the translation of object A by t. Most of the time, the metric represents the Euclidean distance between a pair of objects. For spatial objects a qualitative metric does not necessarily follow this rule.

We will define metrics for topological relations that capture the semantic knowledge about the relation between a pair of objects. These metrics overcome the limitations of topological spatial relations. For example, to measure the part of A split by B (i.e., the part common to A and B), the metric is defined by $m(A, B) = \text{volume}(A \cap B)/\text{volume}(A)$. As such, for this metric m, (1) $m(A, A) = 1$, not zero, *anti-identity*; (2) $m(A, B)$ is not necessarily equal to $m(B, A)$, *anti-symmetric*; and (3) m does not satisfy the triangle inequality, *anti-triangle inequality*. For our purpose this metric provides very useful information to determine the quality of topological connectivity relations (see Section V).

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D. Region Connection Calculus Spatial Relations

Much of the foundational research on qualitative spatial reasoning concerns a region connection calculus (RCC) that describes 2D regions (i.e., topological space) by their possible relations to each other. RCC8 can be formalized by using first order logic [1] or using the 9-intersection model [2]. Conceptually, for any two regions, there are three possibilities: (1) *One object is outside the other*; this results in the RCC8 relation DC (disconnected) or EC (externally connected). (2) *One object overlaps the other across boundaries*; this corresponds to the RCC8 relation PO (proper overlap). (3) *One object is inside the other*; this results in topological relation EQ (equal) or PP (proper part). To make the relations jointly exhaustive and pairwise distinct (JEPD), there is a converse relation denoted by PPc (proper part *converse*), PPc(A,B) ≡ PP(B,A). For completeness, RCC8 decomposes proper part into two relations: TPP (tangential proper part) and NTPP (non-tangential Proper part). Similarly for PPc, RCC8 defines TPPc and NTPPc. The RCC8 relations are pictorially described in Fig. 3.

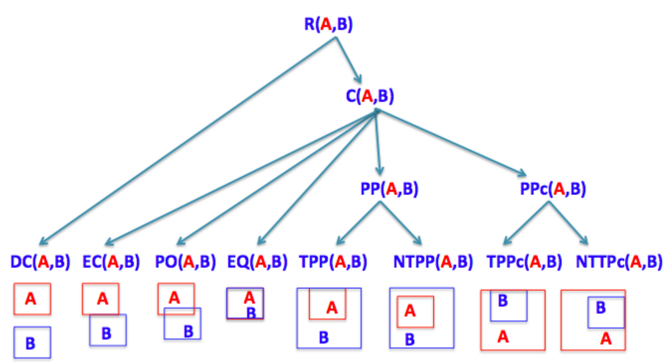

Fig. 3. RCC8 Relations in 2D.

Each of the RCC8 relations can be uniquely described by using a 9-Intersection framework. This is a comprehensive way to look at the relation between two regions. Table 1 depicts the 9-Intersection matrix between two regions A and B, where Int represents the region's interior, Bnd denotes the boundary, and Ext represents the exterior. The predicate IntInt(A,B) is a binary relation that represents the intersection between the interiors of region A and region B; the value of this function is either true (non-empty) or false (empty) for that intersection. Similarly, there are other predicates for the intersection of A's interior, exterior, or boundary with those of B.

For two non-empty bounded regions, A and B, the intersection of their exteriors is always non-empty; it adds nothing to the discrimination and knowledge discovery about regions. In our prior work [7], we have consistently replaced the 9-Intersection 3×3 matrix with the 8-Intersection to define the spatial relations. The values of the 8-Intersection for the RCC8 relations are given below in Table 2.

TABLE 1.
9-INTERSECTION 3×3 MATRIX AND REDUCED 4-INTERSECTION 2×2 MATRIX (SHADED) FOR CALCULATING RCC8 RELATIONS

	Interior	**Boundary**	**Exterior**
Interior	Int(A)∩Int(B)	Int(A)∩Bnd(B)	Int(A)∩Ext(B)
Boundary	Bnd(A)∩Int(B)	Bnd(A)∩Bnd(B)	Bnd(A)∩Ext(B)
Exterior	Ext(A)∩Int(B)	Ext(A)∩Bnd(B)	Ext(A)∩Ext(B)

From careful analysis of Table 2, we see that the IntInt and BndBnd columns have the most useful information in the sense that they are sufficient to partition the RCC8 relations into three classes: {DC, EC}, {NTTP, NTTPc}, and {PO, EQ, TPP, TPPc}.

TABLE 2.
THE VALUES OF THE 8-INTERSECTION VECTORS AND 4-INTERSECTION VECTORS (SHADED) THAT ARE REQUIRED TO DISTINGUISH RCC8 RELATIONS

	IntInt	**BndBnd**	**IntBnd**	**BndInt**	**IntExt**	**BndExt**	**ExtInt**	**ExtBnd**
DC	**F**	**F**	**F**	**F**	**T**	**T**	**T**	**T**
EC	**F**	**T**	**F**	**F**	**T**	**T**	**T**	**T**
NTPP	**T**	**F**	**F**	**T**	**F**	**F**	**T**	**T**
NTPPc	**T**	**F**	**T**	**F**	**T**	**T**	**F**	**F**
EQ	**T**	**T**	**F**	**F**	**F**	**F**	**F**	**F**
TPP	**T**	**T**	**F**	**T**	**F**	**F**	**T**	**T**
TPPc	**T**	**T**	**T**	**F**	**T**	**T**	**F**	**F**
PO	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**

Further analysis of Table 2 indicates that only 4-intersections are sufficient for classification of topological relations [11]. This table can be interpreted and formulated in terms of rules for system integration. These rules are displayed for visualization in the form of a decision tree in Fig. 4.

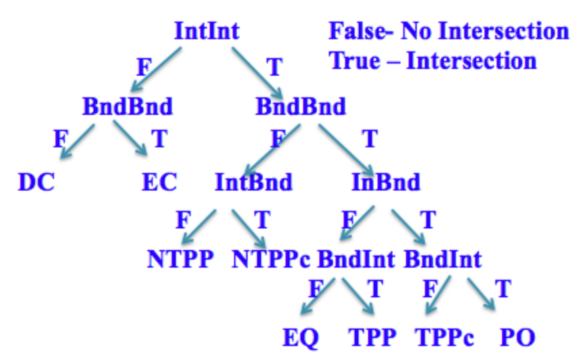

Fig. 4. Tree for classifying the topological relations, (T/F) represents whether or not the objects intersect.

Originally, region connection calculus was designed for 2D [1], [2]; later it was extended to 3D [8], [6]. In [5], metrics were used for associating line-region and region-region connectivity in 2D to natural language. The metrics were adapted from [5] for qualitative study of the dependency

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between metrics and topological relations, and between metrics and natural-language terms; conclusions then were drawn for association between several natural-language terms and the topological connectivity RCC8 terms [7]. However, the 2D metrics were adopted and adapted to 3D objects without any regard for viability or completeness. Herein we introduce new metrics and describe a complete set of metrics to explore the degree of association between them in terms of strength of connectivity and relative size information.

III. Motivation for Metrics

In qualitative spatial reasoning, there are three distinct properties for reasoning about spatial objects: connection, dimension, and direction. Reasoning over *combinations* of these properties can provide additional useful knowledge. The prior efforts [5] have lacked rigorous analysis, expressive power, and completeness of the associated metrics. Revision of the metrics is required before we can begin to bridge the gap between topological connectivity and size information for automated spatial reasoning.

We start with following example for motivation to study the degree (or extent) of spatial relations. This example centers around one metric and one pair of objects; see Fig. 5 for concept illustration. Consider the volume of interior of an object A split by the volume of interior of an object B; let this be denoted by metric, IVsIV(A,B). The definition of metric considers one object, the first parameter, as the reference object and the other object, the second parameter, as the target object. The target object is the object that performs the splitting action on the reference object. This metric calculates how much of reference object A is part of the target object B. Since sizes of objects can vary in units of measurement, it is more reasonable to compare qualitative or relative sizes for objects. Recall from section II.B that A^i represents the interior of A. We define the relative (i.e., normalized) part of A common in B by the equation,

$$IVsIV(A,B) = \frac{volume(A^i \cap B^i)}{volume(A^i)}$$

With this metric, let us see in what ways, the connectivity and size information are useful in spatial reasoning.

(1) *RCC8 topological relation:* Suppose that for objects A and B in Fig. 5, we have IVsIV(B,A) = 1. This implies B is a proper part of A, PP(B,A), which is an RCC8 qualitative connectivity relation. Without the metric, in general, this relation is computed by using the 9-intersection model involving various pairwise intersections before arriving at this conclusion [2], [10]. The metric provides this information much more quickly and efficiently.

(2) *Size relations*: In Fig. 5, suppose IVsIV(A,B) = 0.1, which implies that 10% of A is part of B and part of B is 10% of A. From step (1), IVsIV(B,A) = 1, which means all of B is a part of A. With both the metrics, we conclude that B is equal to 10% of A, and that B is part of A. Therefore, B is much smaller than A for the size relation (i.e., A is much bigger than B). In general, if IVsIV(A,B) < IVsIV(B,A), then A is larger than B in size (i.e., or B is smaller than A in size). Thus the metric is a useful tool for qualitative size comparison of pairs of objects.

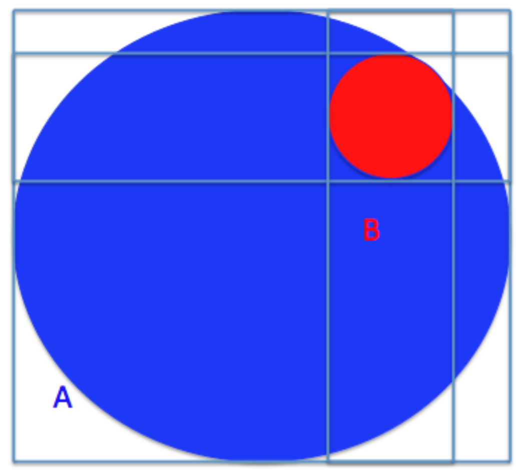

Fig. 5. Object B is a proper part of A, B is much smaller than A in size, and B is in the northeast relative to A.

(3) *Cardinal direction relations:* We will concentrate on steps (1) and (2) in this paper. The grid is generated by grid lines for A and B, where the minimum-bounding rectangle is composed of horizontal and vertical gridlines. The detailed discussion of directions metrics is beyond the scope of this exposition; the reader may consult [9], [10]. The direction metric in [9], [10] determines that B is in the northeast of part of A. With this directional knowledge, it means that in addition to B being a tangential proper part of A, TPP(B,A), tangency is in the northeast direction.

Thus we see that B is a proper part of A, and B is much smaller than A, i.e. B is simply 10% of A. Moreover, B is a tangential proper of A and is located in the northeast part of A.

For an example of the need and usefulness of the metrics, see Section V. Later we will discuss how these metrics measure the degree of connectivity (as shown at the end in Fig. 12), and strengthen the topological classification tree, see Fig. 4.

IV. Introduction to Metrics

With relative movement of objects, their spatial relation change over time. Spatial relations can be used to detect such changes. Shift, erosion and dilation are the most common changes that take place in geographical regions with those changes occurring in a continuous pattern. Quantitative metrics are defined to determine the extent of temporal

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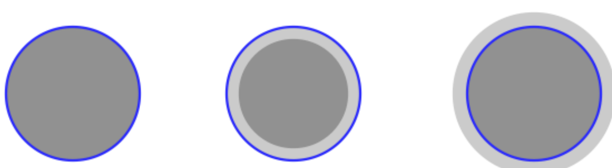
connectivity of the topological relations between pairs of objects in 3D. The metrics are normalized so that the metric values are constrained to [0, 1]. The metrics also facilitate determining the topological relations between objects. As seen in Fig. 5, a metric can be used to derive the qualitative size of the overlap. The overlap relation, PO (A,B), is symmetric, but the overlap metric IVsIV (A,B) is anti-symmetric. The metric values are also sensitive to the location of the objects in addition to topological connectivity, see Fig. 2 and 3.

For the purposes of precisely defining the metrics herein, we will need two additional topological concepts in addition to the traditional interior, boundary, and exterior parts of an object (or region). The classical crisp boundary of an object A is denoted by A^b, see Fig 6(a); for fuzzy regions, the boundary interior neighborhood (Bin) is denoted by A^{bi}, see Fig. 6(b), and the boundary exterior neighborhood (Bex) is denoted by A^{be}, see Fig. 6(c). We give the complete details of these concepts in Section IV.B; an application can selectively use the kind of boundary information available. The exterior and interior boundary neighborhoods even may be combined into one fuzzy/thick boundary which is denoted by A^{bt}, defined as $A^{bt} \equiv A^{bi} \cup A^{be}$, see Fig. 7.



(a) Object boundary | (b) Boundary interior neighborhood | (c) Boundary exterior neighborhood

Fig. 6. (a) A 3D object, (b) the interior neighborhood of the boundary of the object, and (c) the exterior neighborhood of the boundary of the object.

Based on these five region parameters, the 9-Intersection table expands to a 25-Intersection table; see Table 3. For 9-intersection, there are 2^9=512 possible combinations out of which only eight are physically realizable; see Fig. 3. Similarly out of 2^{25} possible combinations derivable from the five region parameters, only a few are physically possible. The possible relations using metrics are as crisp as for bivalent 9-intersection values, see Section V.

A. Volume Considerations

For 3D regions, the volume of a region is a positive quantity, as is the volume enclosed by a cube or a sphere. The classical crisp boundary of a 3D object is 2D, and the volume of a 2D region in a plane or space is zero. Topological relations are predicates that represent the existence of a relation between two objects; metrics measure the strength of the relation or degree of connectivity.

Recall from Section III, the metric IVsIV(A,B) can be used to determine the extent of the overlap $A \cap B$ relative to A, whereas the metric IVsIV(B,A) determines the extent of overlap $A \cap B$ relative to B. For ease and consistency, the metrics are always normalized with respect to the first parameter, of the metric function. The metric IVsIV(A,B) is *anti-symmetric*. It represents the amount of overlap relative to first argument of the metric.

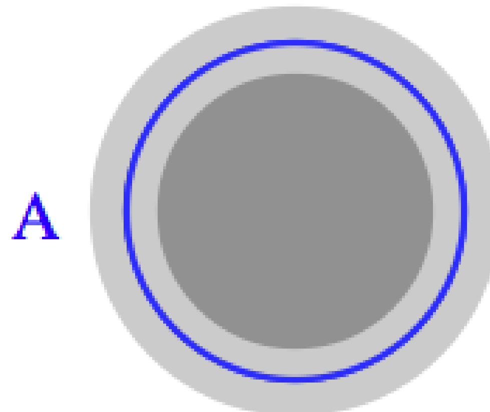


Fig. 7. The space relative to object A is partitioned into 5 parts: interior (dark), A^i; interior-neighborhood (light dark inside the boundary), A^{bi}; boundary, A^b; exterior neighborhood (light dark outside the boundary), A^{be}; and exterior (all outside), A^e

For practical applications, the first parameter is never the exterior volume of an object, because the exterior of a bounded object is unbounded with infinite volume. Also it is observed that since volume(A) = volume($A \cap B^i$) + volume($A \cap B^e$), then IVsIV(A,B^e) = 1- IVsIV(A,B^i).

B. Boundary Considerations

The boundary neighborhood is the region within some small positive radius of the boundary. This is useful for regions with a vague boundary. There are two types of neighborhoods, the boundary interior neighborhood, A^{bi}, and the boundary exterior neighborhood, A^{be}; see Fig. 6. By combining the two, we can create a thick boundary for vague regions; see Fig. 7.

Several metrics are designed for cases where the boundary is vague; these are discussed in Sections IV.F.1 and IV.F.2. To compensate for an accurate crisp boundary, an application-dependent small neighborhood is used to account for the thickness of the boundary. For the 3D object shown in Fig. 6(a), let the boundary interior neighborhood of A^b of some radius r>0, be denoted by A^{bi} or $N_{Ir}(A^b)$, i.e., $A^{bi} \equiv N_{Ir}(A^b)$ (Fig. 6(b)), and let the boundary exterior neighborhood of A^b of some radius r>0, be denoted by A^{be} or $N_{Er}(A^b)$, i.e., $A^{be} \equiv N_{Er}(A^b)$; see Fig. 6(c). The smaller the value of r, the less the ambiguity in the object boundary. Fig. 7 depicts how these terms apply to space partitioning. We denote the qualitative interior neighborhood by $\Delta_I A$ and exterior neighborhood by

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$\Delta_E A$ without specific reference to r, as $\Delta_I A \equiv A^{bi}$ and $\Delta_E A \equiv A^{be}$ in the equations that follow in this paper.

Many times in geographical information system (GIS) applications the region's exact boundary is not available. Thus the problem in spatial domains becomes that of how to identify and represent these objects. In such analyses, the external connectedness would be resolved by using a metric that considers the boundary exterior neighborhoods, BexsBex, and examining whether the value BexsBex (A,B) < min (r_1,r_2) (instead of BsB (A,B)=0, which only considers the crisp boundaries) where the objects have boundary exterior r_1- and r_2-neighborhoods for thick boundaries of objects. For definitions of these metrics see Sections IV.F.1 and IV.F.2.

In fact, some applications may need only one r-neighborhood (the combination of r_1-interior and r_2-exterior neighborhood along a vague boundary), while others may need two separate neighborhoods as in [5]. The value of r = min(r_1,r_2) is specified by the application. For some applications (e.g. numerical calculations) it is approximately one percent of the sum of the radii of two spheres. Intuitively, r accounts for the minimum thickness of the boundary for the vague region. In other applications, in order for the metrics to be useful, the radial distance r is chosen to be equal to distance of boundary of object A from the boundary of the object B, i.e. r_{ab} = dist(A^b, B^b) = min{dist(x,y): $x \in A^b$ and $y \in B^b$}.

C. Intersections in General

All the metrics and topological relations involve intersections (see Table 3) between a pair of objects. An intersection between a pair of objects may be interior to interior (i.e., 3D), or boundary to boundary (neighborhood), which may be turn out to be 2D, or 1D or even 0D. Metrics measure the quantitative values for topological relations. The intersection of 3D objects may remain 3D, as in the case of PO(A,B). If the intersection such as $A^i \cap B^i$ exists, then we can calculate the volume of the 3D intersection $A^i \cap B^i$, which is practical. But if the boundary is 2D, the volume of the boundary is zero, which does not provide any useful information. The intersection between two 3D objects may also be 3D, 2D, 1D, or even 0D, see Fig. 8. Since intersection is a significant component of topological relations, we can extract useful information from intersections of lower dimensional components also. We can calculate the area of a 2D object (e.g., $A \cap B^b$ may be a 2D surface), and surface area can provide essential information for relations EC(A,B), TPP(A,B), and TPPc(A,B). For example, if two cubes touch face to face, they intersect in a surface; the volume of intersection will be zero, but surface area will be positive, which can still provide a measure of how close the objects are to each other. So we will need metrics that accommodate 2D surface area also. Sometimes intersection is a curve or a line segment, in which case we can analyze the strength of the relation from the length of the segment. Consequently, we also need metrics that handle the length of edge intersection. For a single point intersection (degenerate line segment), the volume of a point is zero, as are the area and length of a single point, see Fig. 8.

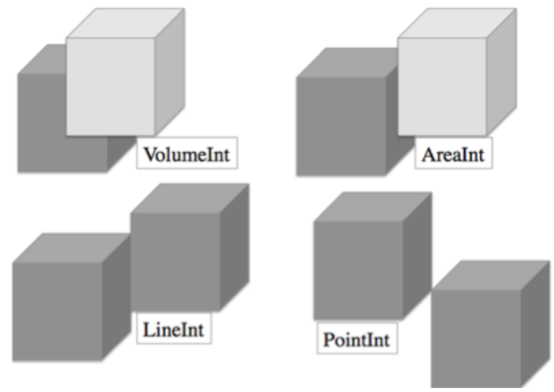

Fig. 8. The intersection two 3D objects can be a point 0D, a line segment 1D, surface area 2D, or volume 3D.

D. Space Partitioning

Usually, each object divides the 3D space into three parts: interior, boundary and exterior. In reality, it is five parts, see Fig. 7. The interior and exterior of the object are 3D parts of space, and the boundary of the object is 2D. The intersection between two 3D objects can be 3D, or a 2D surface, or a 1D curve, or a line segment, or even 0D (i.e., a point). In many geographical applications, regions may not have a well-defined boundary. For example, the shoreline boundary of a lake is not fixed. If the lake is surrounded with a road, the road can serve as the boundary for practical purposes. We need to compensate for the blur in the boundary. Consequently we utilize two additional topological regions: Boundary inner neighborhood (Bin) and Boundary exterior neighborhood (Bex). They can be used to measure how close the objects are from boundary to boundary. The thick boundary becomes a 3D object rather than a 2D object, so the volume calculation for boundary becomes meaningful. For non-intersecting objects, it can be used to account for the distance between them, and for the tangential proper part relation between objects A and B, NTPP(A, B), it can measure how close the inner object A is from the object inner boundary B^{bi}. Thus the terms Boundary interior neighborhood (Bin) and Boundary exterior neighborhood (Bex) for an object A account for the fuzziness, $A^{bt} \equiv A^{bi} \cup A^{be}$, in the boundary description or the thickness of the boundary; see Fig. 6 and Fig. 7.

E. 25-Intersection

To keep full generality available to the end-user, an object space can be defined in terms of five parts: interior, boundary, exterior, boundary interior neighborhood, and boundary exterior neighborhood. As descriptive as we can be for

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symbols to be close to natural language: we use Int(A) for A^i the interior of A, Ext(A) for A^e the exterior of A, Bnd(A) for A^b the boundary of A, Bin(A) for A^{bi} the boundary interior neighborhood A, and Bex(A) for A^{be} the boundary exterior neighborhood of the boundary of A. This will lead to a 25-intersection table where the boundary can be a crisp boundary A^b, or a thick boundary $A^{bt} \equiv A^{bi} \cup A^{be}$; see Table 3 for all 25 combinations of intersections.

TABLE 3
25-INTERSECTION TABLE.

	Int	Bnd	Ext	Bin	Bex
Int	$A^i \cap B^i$	$A^i \cap B^b$	$A^i \cap B^e$	$A^i \cap B^{bi}$	$A^i \cap B^{be}$
Bnd	$A^b \cap B^i$	$A^b \cap B^b$	$A^b \cap B^e$	$A^b \cap B^{bi}$	$A^b \cap B^{be}$
Ext	$A^e \cap B^i$	$A^e \cap B^b$	$A^e \cap B^e$	$A^e \cap B^{bi}$	$A^e \cap B^{be}$
Bin	$A^{bi} \cap B^i$	$A^{bi} \cap B^b$	$A^{bi} \cap B^e$	$A^{bi} \cap B^{bi}$	$A^{bi} \cap B^{be}$
Bex	$A^{be} \cap B^i$	$A^{be} \cap B^b$	$A^{be} \cap B^e$	$A^{be} \cap B^{bi}$	$A^{be} \cap B^{be}$

Now Bnd(A) represents the crisp boundary of A, if any, whereas Bin(A) and Bex(A) account for the crisp representations of the vague boundary. There are 2^{25} possible 25-intersection vectors in all. However, all the vectors are not physically realizable. For example, all entries in any row in Table 3 cannot be true simultaneously, and all entries in any column in Table 3 cannot be true simultaneously. Another use of the metrics is to see, for the proper part relation between A and B, PP(A,B), how far the inner object A is from the inner boundary neighborhood of the enclosing object, B^{bi}. A commonly used predicate for determining *weak* connectivity between crisp regions is boundary-boundary intersection, $A^b \cap B^b$. We must be mindful that space now is portioned into five parts instead of three parts. It is clear that A^i, A^e are open sets, and A^b is a closed set. For spatial reasoning, when A^{bi} and A^{be} are used, they are semi-open, semi-closed sets — open towards A^b and closed towards the inside of A^{bi} and the outside of A^{be}.

F. Developing Spatial Metrics

Here we complete the development of the remaining metrics; an application may selectively use the metrics applicable to the problem at hand. If IVsIV(A,B)=0 and BexsIV(A,B)=0, it means that A,B are at least r = d_{ab} (distance between the exterior neighborhood of A and the boundary of B). Conventionally, a 4-intersection [10] (BndBnd, IntBnd, BndInt, IntInt) is sufficient to represent crisp 3D data. Some applications may represent Bex and Bin separately [5], while fuzzy logic applications may need to combine Bex and Bin into Bnd [6]. For all 25 intersections (see Table 3) the metrics are defined by normalizing the intersections. There are 25 possible pairwise intersections to be considered in the metrics. For one pair of objects, consider the eight distinct versions {(A,B), (A,B^e), (A^e,B), (A^e,B^e), (B,A), (B,A^e), (B^e,A), (B^e,A^e)} as input arguments for which a metric value may be computed. That is, the domain for each metric consists of eight distinct pairs corresponding to each input pair of objects A and B. Since metrics are normalized, some metrics may not be realizable; for example, IVsIV cannot be defined for the combinations {(A^e,B), (A^e,B^e), (B^e,A), (B^e,A^e)} because the corresponding metrics involve infinity. In fact, five of the metrics are impossible (not realizable); see Table 4. Here we will identify the possible (reasonable) 20 metrics.

TABLE 4
Complete list of metrics corresponding to 25 intersections in table 1. 20 metrics are viable and 5 metrics are not possible.

Possible	Impossible
IVsIV, IVsEV	EVsIV, EVsEV
BinsIV, BinsEV, IVsBin	EVsBin
BexsIV, BexsEV, IVsBex	EVsBex
BinsBin, BinsBex,	
BexsBin, BexBex	
BsIV, BsEV, IVsB	EVsB
BsBin,BsBex,	
BinsB, BexsB	
BsB	

Since the metrics are anti-symmetric, the converse metrics can be defined by switching arguments A and B (e.g., the converse of IVsIV(A,B) is IVsIV(B,A)). To make the list of metrics exhaustive, we can append suffix c to the name to indicate the converse metric when needed. Table 2 lists directly possible and impossible metrics, which are further developed in detail in Sections IV.F.1 and IV.F.2.

Next we will define 20 viable metrics and show their connection with the RCC8 topological relations and size relations on 3D objects only. First we look at the two metrics together: IVsIV(A,B) and IVsEV(A,B) which measure how much space one object shares with the other object. We have already defined interior volume split by interior volume, IVsIV(A,B), earlier in the motivation discussion, Section III.

F.1. Volume Metrics: segmentation of interior of reference object by the interior and exterior of target object

Recall, interior volume splitting (IVsIV) measures the scaled (normalized) part of reference object that is split by the interior of the target object. It calculates how much of A is part of B. The crisp boundary of a 3D object is 2D. Here boundary does not matter, as the volume of the boundary is zero. Exterior volume splitting (IVsEV) describes the proportion of reference object's interior that is split by the other target object's exterior. The exterior volume splitting metric (IVsEV) is defined by

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$$IVsEV(A,B) = \frac{volume(A^i \cap B^e)}{volume(A^i)}$$

It measures how much A is away from B. Again, boundary does not matter. Observe that volume(A) = volume(A∩B) + volume($A \cap B^e$), and hence IVsEV(A,B) = 1 - IVsIV(A,B). The metric value is between 0 and 1, inclusive. If the metric value IVsIV(A,B) = 0, the objects are disjoint or externally connected. If the metric value IVsIV(A,B) > 0, then this value indicates two things. First, $A^i \cap B^i \neq \emptyset$. Usually, the truth value of $A^i \cap B^i$ is established by considering the intersection of the boundaries of two objects (extensive computation takes place because the objects are represented with boundary information only). Here if the metric value IVsIV(A,B) > 0, we can quickly determine the truth value of $A^i \cap B^i$. Secondly, the actual value of the metric IVsIV(A,B) measures what relative portion of object A is common with object B; the larger the value of the metric, the larger the commonality and conversely. Let

$$x = \frac{volume(A^i \cap B^i)}{volume(A^i)} * 100 \quad y = \frac{volume(B^i \cap A^i)}{volume(B^i)} * 100$$

This can directly answer queries such as object A has x percent in common with B, whereas object B has y percent in common with A, provided A and B intersect. If x=y=0, then the objects are either externally connected or disjoint, but this metric alone does not tell how far apart they are. In order to determine that, we simply compute the distance between the boundaries to differentiate between DC and EC. The metric does embody knowledge about which object is larger. If x<y, then object A is bigger than object B.

F.2. Boundary Metrics: segmentation of thick boundary of reference object by the interior and thick boundary of target object

Recall, for the 3D object shown in Fig. 6(a), A^{be} is the boundary exterior neighborhood of A^b with some radius (Fig. 6(b)), and A^{bi} is the boundary interior neighborhood of A^b with some radius, see Fig. 6(c). The value of the radius is application-dependent. We use the qualitative interior and exterior neighborhood without specific reference to r, as $\Delta_I A \equiv A^{bi}$ and $\Delta_E A \equiv A^{be}$ in the following equations.

Considering the interior neighborhood of the reference object, we define the closeness to interior volume (BinsIV) as follows:

$$BinsIV(A,B) = \frac{volume(\Delta_I A \cap B^i)}{volume(\Delta_I A)}$$

This metric contributes to the overall degree of relations of PO, EQ, TPP, and TPPc.

Similarly, we can consider the exterior neighborhood of an object, and can define a metric for exterior volume closeness (*BexsIV*) by replacing $\Delta_I A$ by $\Delta_E A$.

$$BexsIV(A,B) = \frac{volume(\Delta_E A \cap B^i)}{volume(\Delta_E A)}$$

This metric is a measure of how much of the exterior neighborhood of A^b is aligned with the interior of B. This metric is useful for analyzing the degree of relations of PO, EQ, TPP, and TPPc.

Similarly the metrics for the exterior of B are defined for completeness as follows:

BeinsEV(A,B) becomes

$$BinsEV(A,B) = \frac{volume(\Delta_I A \cap B^e)}{volume(\Delta_I A)}$$

BexsEV(A,B) is defined by replacing $\Delta_I A$ by $\Delta_E A$.

$$BexsEV(A,B) = \frac{volume(\Delta_E A \cap B^e)}{volume(\Delta_E A)}$$

Boundary-boundary intersection is an integral predicate for distinguishing RCC8 relations.

Similarly, for quantitative metrics, it can be important to consider how much of the inside and outside of the boundary neighborhood of reference object is shared with the boundary neighborhood of the other object.

BinsBin(A,B) is designed to measure how much of the Interior Neighborhood of A is split by the Interior Neighborhood of B. This metric is useful for fuzzy regions with fuzzy interior boundary.

$$BinsBin(A,B) = \frac{volume(\Delta_I A \cap \Delta_I B)}{volume(\Delta_I A)}$$

BexsBin(A,B) is designed to measure how much of the Exterior Neighborhood of A is split by the Interior Neighborhood of B.

$$BexsBin(A,B) = \frac{volume(\Delta_E A \cap \Delta_I B)}{volume(\Delta_E A)}$$

This metric is useful when the region is vague around both sides of the boundary.

BinsBex(A,B) is defined by replacing $\Delta_I A$ by $\Delta_E A$ and is designed to measure how much of the interior neighborhood of A is split by the exterior neighborhood of B, It is useful to analyze topological relations DC and EC.

$$BinsBex(A,B) = \frac{volume(\Delta_I A \cap \Delta_E B)}{volume(\Delta_I A)}$$

BexsBex(A,B) is designed to measure how much of the exterior neighborhood of A is split by the exterior neighborhood of B. This metric is useful for fuzzy regions, if BexsBex(A,B) = 0 then we can narrow down the candidates of possible relations between A and B to DC, NTPP, and NTPPc.

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$$BexsBex(A,B) = \frac{volume(\Delta_E A \cap \Delta_E B)}{volume(\Delta_E A)}$$

F.3. Boundary Metrics: segmentation of crisp boundary of reference object by the interior and the boundary of target object

We define several splitting metrics to specifically examine the proportion of the boundary of the reference object that is split by the volume, boundary neighborhoods, and boundary of the target object; we denote these metrics accordingly for boundary splitting. It should be noted that there are five versions of the equations for this metric. First, the boundary may be the thick boundary composite neighborhood (interior and exterior), in which case it is a volume, see Fig. 8. If the boundary is a simple boundary, it is a 2D area. Therefore, for numerator calculations, we will be calculating $A^b \cap B$ as either a volume or an area. It also is possible that $A^b \cap B$ is an edge (a curve or a line segment). For example, for two cubes, a cube edge may intersect the face of the cube as a line segment or an edge of another cube in a line segment, or even as a single point (i.e., a degenerate line segment). If $A^b \cap B$ is an edge, we calculate edge length. For the denominator, volume(A^b) and area(A^b) are self-evident depending on whether we have a thick or simple boundary. However, length(A^b) calls for an explanation. In the numerator, when length($A^b \cap B$) is applicable, then this intersection is part of an edge in A^b; length(A^b) is computed as the length of the enclosing edge. These metrics are defined and described below. The converses of the metrics can be derived similarly.

BsIV(A,B) measures the Boundary of A split by the Interior Volume of B.

$$BsIV(A,B) = \frac{volume(A^b \cap B^i)}{volume(A^b)} \text{ or } \frac{area(A^b \cap B^i)}{area(A^b)} \text{ or } \frac{length(A^b \cap B^i)}{length(A^b)}$$

BsEV(A,B) is defined by replacing B^i by B^e and measures the Boundary of A split by the Exterior Volume of B.

$$BsEV(A,B) = \frac{volume(A^b \cap B^e)}{volume(A^b)} \text{ or } \frac{area(A^b \cap B^e)}{area(A^b)} \text{ or } \frac{length(A^b \cap B^e)}{length(A^b)}$$

BsBin(A,B) is defined by replacing B^i by $\Delta_I(B)$ and measures the Boundary of A split by the Interior Neighborhood of B.

$$BsBin(A,B) = \frac{volume(A^b \cap \Delta_I B)}{volume(A^b)} \text{ or } \frac{area(A^b \cap \Delta_I B)}{area(A^b)} \text{ or } \frac{length(A^b \cap \Delta_I B)}{length(A^b)}$$

BsBex(A,B) is defined by replacing B^i by $\Delta_E B$ and measures the Boundary of A split by the Exterior Neighborhood of B.

$$BsBex(A,B) = \frac{volume(A^b \cap \Delta_E B)}{volume(A^b)} \text{ or } \frac{area(A^b \cap \Delta_E B)}{area(A^b)} \text{ or } \frac{length(A^b \cap \Delta_E B)}{length(A^b)}$$

BsB(A,B) is defined by replacing B^i by B^b and measures the Boundary of A split by the Boundary of B.

$$BsB(A,B) = \frac{volume(A^b \cap B^b)}{volume(A^b)} \text{ or } \frac{area(A^b \cap B^b)}{area(A^b)} \text{ or } \frac{length(A^b \cap B^b)}{length(A^b)}$$

This metric is again directly applicable to computing $A^b \cap B^b$ which is used to distinguish many of the RCC8 relations. This subsequently allows us to narrow down the candidates of possible relations between A and B to DC, NTPP, and NTPPc.

For crisp regions, we have an interior, boundary, and exterior. For vague regions, we have boundary interior and exterior neighborhoods. The smaller the radius for boundary neighborhoods, the smaller the ambiguity in the object boundary. For consistency, we can combine the interior and exterior neighborhoods into one, which we call a thick boundary. For a thick boundary, the object has three disjoint crisp parts: the interior, the thick boundary, and the exterior. Now we can reason with these parts similar to how we use crisp regions for determining the spatial relations.

F.4. Anatomy of Metrics

The following discussion is applicable when there is positive distance between the boundaries, $A^b \cap B^b = \emptyset$, irrespective of $A^i \cap B^i \neq \emptyset$, see Fig. 9, or $A^i \cap B^i = \emptyset$, see Fig. 10. In Fig. 7, we see that the space relative to an object A can be partitioned into 5 parts: interior (dark), interior-neighborhood (light dark inside the boundary), boundary, exterior neighborhood (light dark), and exterior (all outside). In Fig 9, B shares only A^i. In Fig 10, B shares only A^e. In Fig 11, B shares all five parts, A^i, A^{bi}, A^b, A^{be}, and A^e.

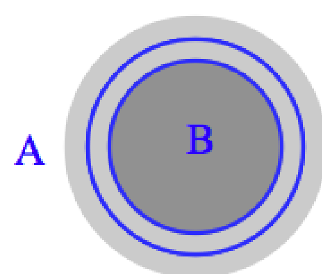

Fig. 9. The space relative to object A is partitioned into 5 parts: interior (dark), interior-neighborhood (light dark inside the boundary), boundary, exterior neighborhood (light dark), and exterior (all outside). A and B have disjoint boundaries, B is inside A, but away from the boundary by $r = d_{ab}$.

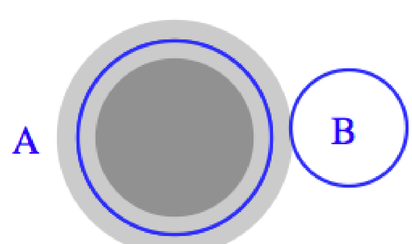

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Fig. 10. The space relative to object A is partitioned into 5 parts: interior (dark), interior-neighborhood (light dark inside the boundary), boundary, exterior neighborhood (light dark), and exterior (all outside). A and B have disjoint boundaries, B is outside A ,but away from the boundary by $r = d_{ab}$.

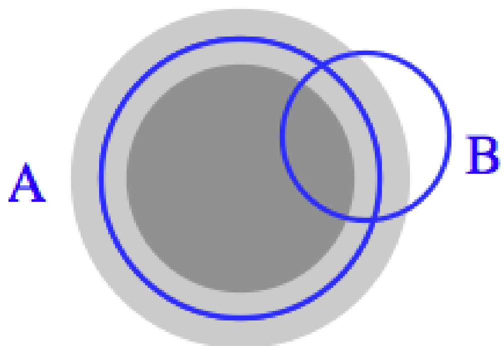

Fig. 11. An example where B shares A^i, $\Delta_I A \equiv A^{bi}$, A^b, $\Delta_E A \equiv A^{be}$, and A^e.

V. Spatial Relations: Topological and Metrical

If the regions are crisp, we can use the 9-intersection model for determining connectivity relations for 2D connectivity knowledge [2],[10], and for relative size information we use the 3D metrics from Section IV. The relative size of the objects and boundary is obtained by using volume metrics IVsIV, IVsB, and boundary-related BsB metrics. Metrics measure the degree of connectivity; for example, for the proper overlap relation PO(A,B), IVsIV metric helps to determine the relative extent of overlap of each object. In Section IV we discussed which metrics are specific to each of the connectivity relations. If one or both regions have vague boundaries, we can use metrics to create a thick boundary, $A^{bt} \equiv A^{bi} \cup A^{be}$, by using the interior and exterior neighborhoods. Again we have, crisp interior A^i, exterior A^e, and thick boundary A^{bt}.

By using the 9-Intersection model on A^i, A^e, and A^{bt}, we can derive the connectivity, degree of connectivity, and relative size information for vague regions. Other applications such as natural language and topological association [5] can use appropriate combinations of these topological parts. Fig. 5 provides a visual summary of: (1) what intersections are required to classify each topological relation, and (2) the contribution (T / F) indication if the intersection is non-empty or empty. This tree can be used to classify crisp topological relations. Fig. 12 provides a visual summary of: (1) what metrics are required to classify each topological relation, and (2) the contribution (0 / +) each metric has with regards to the overall quality of the relation. This tree can be used to classify crisp relations. Similarly, a tree could be generated for vague regions with appropriate metrics from the set of 20 metrics.

The general frame work for combining topology and metrics given in Fig. 12 can be explained diagrammatically in Fig. 13 as follows. First of all topological relations are determined. Then metrics are developed to strengthen the quality of the topological relations. Finally the metric based tree is generated that captures the topological and metrical relations in a single hybrid tree.

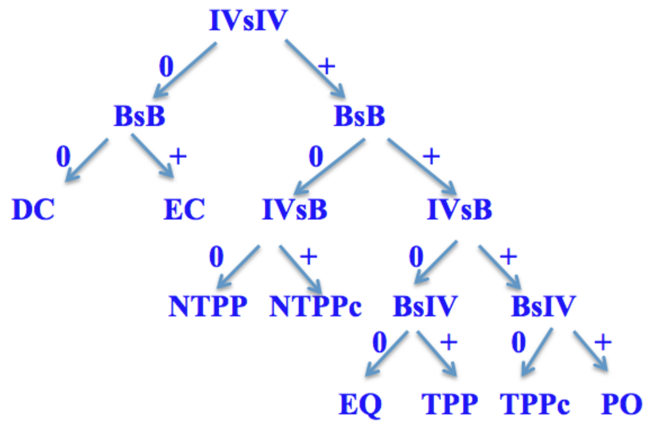

Fig. 12. Tree for the metrics required for classification and the contribution (0/+) of the respective metrics to the overall quality of classification.

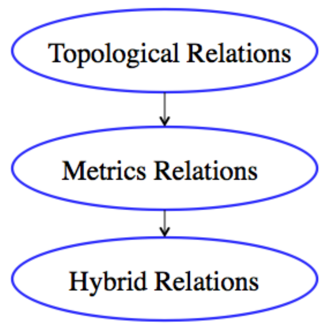

Fig. 13. The structure of topological relations system and metrics relations leading to hybrid system.

VI. Conclusions and Future Directions

Herein we presented an exhaustive set of metrics for use with both crisp and vague regions, and showed how each metric is linked to RCC8 relations for 3D objects. Our metrics are systematically defined and are more expressive (consistent with natural language) than previously published efforts. Further, we showed the association between our metrics and the topology and size of objects.

This work should be useful for a variety of applications dealing with automated spatial reasoning in 3D. In the future, we plan to use these metrics to associate natural language terminology with 3D region connection calculus including occlusion considerations. Also we will explore the applications of these metrics between heterogeneous dimension objects, $O_m \in R^m$ and $O_n \in R^n$ for m, n $\in \{1,2,3\}$.

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