ISSN 2007-9737

# A 4-Valued Logic that Extends the Paraconsistent Logic G'3

Mauricio Osorio[1], José Luis Carballido[2]

[1] Universidad de las Américas Puebla,
Puebla,
Mexico

[2] Benemérita Universidad Autónoma de Puebla,
Puebla,
México

{osoriomauri, jlcarballido7}@gmail.com

**Abstract.** We introduce a new 4-valued logic that we call $M4M4$. We show that $M4M4$ is conservative extension of the 3-valued logic $G'_3$, which serves as the formalism to define the p-stable semantics of logic programming. $M4M4$ possesses two negation operators. The weak negation operator that corresponds to the negation operator of $G'_3$. In addition, $M4M4$ also includes a strong negation operator that is the new feature of this logic with respect to $G'_3$. It is well known that allowing these two negations is very useful in knowledge representation. $M4M4$ can be used as the formalism to define the p-stable semantics as well as the stable semantics. We also present other suitable properties of $M4M4$.

**Keywords.** Knowledge representation, stable semantics, logic programming.

## 1 Introduction

Deductive databases are an important aspect in the convergence of artificial intelligence and databases [6].

Currently it is necessary to have complex reasoning tasks to deal with great amounts of data. Logic based systems are an option to provide such complex reasoning capabilities.

Specifically, Deductive Database Systems are forms of database management systems whose storage structures are designed around a logical model of data and at the same time, inference modules for the Deductive Database Systems are designed on logic programming systems.

The Deductive Database Systems are based on deductive database theories that always have associated a semantics. In general, a deductive database theory may give different answers to a query depending on the semantics used.

Two of the semantics that a database theory can be based on, are the stable logic programming semantics (stable semantics) as well as the p-stable logic programing semantics (p-stable semantics).

The mathematical formalism to support those semantics is the theory of intermediate and paraconsistent logics; thus, intuitionism helps to express the stable semantics and the logic $G'_3$ helps to express the p-stable semantics.

In this work we study a new 4-valued logic called $M4M4$. $M4M4$ has a strong negation besides having the native negation operator. We prove that $M4M4$ is a conservative extension of $G'_3$.

Furthermore, $M4M4$ can be used as the formalism to extend the p-stable semantics to a version that includes strong negation in a similar way as the stable semantics has been extended to include such a negation [4].

Our paper is structured as follows. In section 2, we summarize some definitions and logics necessary to understand this paper. In section 3, we introduce the new $M4M4$ logic that is a conservative extension of logic $G'_3$.

This new logic satisfies a substitution theorem, and can express the stable semantics as well as the p-stable semantics. Finally, in section 4, we present some conclusions.

## 2 Background

In this section we summarize some basic concepts and definitions necessary to understand this paper.

### 2.1 Logics

We present several logics that are useful to define and study the new $M4M4$ logic. We assume that the reader has some familiarity with basic logic such as chapter one in [3].

#### 2.1.1 Hilbert Style Proof Systems

One way of defining a logic is by means of a set of axioms together with the inference rule of Modus Ponens. As examples we offer two important logics defined in terms of axioms, which are related to the logics we study later. $C_\omega$ logic [1] is defined by the following set of axioms:

1. $\alpha \to (\beta \to \alpha)$ ,
2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$,
3. $\alpha \wedge \beta \to \alpha$,
4. $\alpha \wedge \beta \to \beta$,
5. $\alpha \to (\beta \to (\alpha \wedge \beta))$,
6. $\alpha \to (\alpha \vee \beta)$,
7. $\beta \to (\alpha \vee \beta)$,
8. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$,
9. $\alpha \vee \neg\alpha$,
10. $\neg\neg\alpha \to \alpha$.

The first eight axioms of the list define positive logic. Note that these axioms somewhat constraint the meaning of the $\to$, $\wedge$ and $\vee$ connectives to match our usual intuition.

It is a well known result that in any logic satisfying axioms Pos1 and Pos2, and with modus ponens as its unique inference rule, the Deduction Theorem holds [3]. We present a Hilbert-style axiomatization of $G'_3$ that is a slight (equivalent) variant of the one presented in [5].

**Table 1.** Truth tables: connectives in $\mathrm{G}_3$ and $\mathrm{G}'_3$

| $x$ | $\neg_{\mathrm{G}_3} x$ | $\neg_{\mathrm{G}'_3} x$ |
|---|---|---|
| 0 | 2 | 2 |
| 1 | 0 | 2 |
| 2 | 0 | 0 |

| $\to$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 2 | 2 | 2 |
| 1 | 0 | 2 | 2 |
| 2 | 0 | 1 | 2 |

We present this logic, since it will be extended to a new logic called $M4M4$, which possesses a strong negation and is the main contribution of this work.

$M4M4$ logic has five primitive logical connectives, namely GPC := $\to, \wedge, \vee, \neg, \sim$. $M4M4$-formulas are formulas built from these connectives in the standard form. We also have two defined connectives:

– $\backsim \alpha := \alpha \to (\neg\alpha \wedge \neg\neg\alpha)$,
– $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$ .

GLukG logic has all the axioms of $C_\omega$ logic plus the following:

– E1. $(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$,
– E2. $\neg\neg(\alpha \to \beta) \leftrightarrow ((\alpha \to \beta) \wedge (\neg\neg\alpha \to \neg\neg\beta))$,
– E3. $\neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$,
– E4. $(\beta\wedge \sim \beta \to (\sim\sim \alpha \to \alpha)$ ,
– E5. $\neg\neg(\alpha \vee \beta) \leftrightarrow (\neg\neg\alpha \vee \neg\neg\beta)$.

Note that Classical logic is obtained from GLukG by adding to the list of axioms any of the following formulas: $\alpha \to \neg\neg\alpha$, $\alpha \to (\neg\alpha \to \beta)$, $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.

On the other hand, $\sim \alpha \to \neg\alpha$ is a theorem in GLukG, that is why we call the $\sim$ connective a non-native strong negation operator.

In this paper we consider the standard substitution, here represented with the usual notation: $\phi[\alpha/p]$ is $\alpha$ when will denote the formula that results from substituting the formula $\alpha$ in place of the atom p, wherever it occurs in $\phi$.

Recall the recursive definition: if $\phi$ is atomic, then $\phi[\alpha/p]$ is $\alpha$ when $\phi$ equals $p$, and $\phi$ otherwise. Inductively, if $\phi$ is a formula $\phi_1 \Box \phi_2$ , for any binary connective $\Box$. Finally, if $\phi$ is a formula of the form $\neg\phi_1$ , then $\phi[\alpha/p]$ is $\neg\phi_1[\alpha/p]$.

**Table 2.** Truth table of the implication in $M4M4$

| $\to$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 3 | 3 | 3 | 3 |
| 1 | 3 | 3 | 3 | 3 |
| 2 | 0 | 1 | 3 | 3 |
| 3 | 0 | 1 | 2 | 3 |

### 2.1.2 GLukG as a Multi-Valued Logic

It is very important for the purposes of this work to note that GLukG can also be presented as a multi-valued logic. Such presentation is given in [2], where GLukG is called $G'_3$.

In this form it is defined through a 3-valued logic with truth values in the domain D = 0, 1, 2 where 2 is the designated value.

The evaluation functions of the logic connectives are then defined as follows: $x \wedge y = \min(x, y), x \vee y = \max(x, y)$; and the $\neg$ and $\to$ connectives are defined according to the truth tables given in Table 1.

We write $\vDash \alpha$ to denote that the formula $\alpha$ is a tautology,namely that $\alpha$ evaluates to 2 (the designated value) for every valuation.

In this paper we keep the notation $G'_3$ to refer to the multi-valued logic just defined, and we use the notation GLukG to refer to the Hilbert system defined at the beginning of this section.

There is a small difference between the definitions of $G'_3$ and Gödel logic $G3$: the truth value assigned to $\neg 1$ is 0 in $G3$. $G3$ accepts an axiomatization that includes all of the axioms of intuitionistic logic.

In particular, the formula $(\alpha \wedge \neg\alpha) \to \beta$ is a theorem in $G3$ but is not a theorem in $G'_3$. The next couple of results are facts we already know about the logic $G'_3$.

**Theorem 1.** [5] For every formula $\alpha$, $\alpha$ is a tautology in $G'_3$ iff $\alpha$ is a theorem in GLukG.

**Theorem 2.** (Substitution theorem for $G'_3$ logic). [5] Let $\alpha, \beta$ and $\Psi$ be GLukG-formulas and let $p$ be an atom. If $\alpha \leftrightarrow \beta$ is a tautology in $G'_3$ then $\Psi[\alpha/p] \leftrightarrow \Psi[\beta/p]$ is a tautology in $G'_3$.

**Corollary 1.** [5] Let $\alpha, \beta and \psi$ be GLukG-formulas and let $p$ be an atom. If $\alpha \leftrightarrow \beta$ is a theorem in GLukG then $\Psi[\alpha/p] \leftrightarrow \Psi[\beta/p]$ is a theorem in GLukG.

## 3 The Logic M4M4

Next we introduce the following 4-valued logic, which we will call $M4M4$. It counts with three negations, one of them is defined as Łukasiewicz negation (in Ł$_4$),denoted here as $\sim x$ which becomes the strong negation.

The weak or standard negation is denoted by $\neg$ and defined as follows: $\neg 3 = 0$ and $\neg x = 3$ for any value different from zero.

The third negation is denoted by the symbol $\backsim$ and is defined as $\backsim x = x \to 0$, where the connective implication is defined below.

The double implication is defined as usual, as the conjunction of two opposite conditionals. There is only one designated value, which is 3.

With the three negations logic $M4M4$ counts with six connectives: conjunction and disjunction are binary connectives and are defined as the minimum and maximum of the two values respectively. The implication is a binary connective defined by Table 2.

We define the bottom particle by the formula $\bot = \neg x \wedge \neg\neg x$ for any atom $x$.

As mentioned before the first interesting property of $M4M4$ logic that relates to our previous sections, is the fact that logic $G'_3$ is expressed in terms of it.

**Theorem 3.** The restriction of logic $M4M4$ to the three values 0, 2, 3 and its weak negation coincides with logic $G'_3$. In particular if $\vDash_{M4M4} \alpha$ then $\vDash_{G'_3} \alpha$.

**Proof.** It is enough to observe that the connectives of $G'_3$ and those of $M4M4$ when restricted to the values 0, 2, 3 have the same truth tables if we interpret the values 2 and 3 of $M4M4$ as the values 1 and 2 of $G'_3$ respectively.

In fact, we have a converse for the second part of the previous result, according to which, we obtain equivalence between logics $M4M4$ and $G'_3$ as established in next result.

**Theorem 4.** For any formula $\alpha$, if $\models_{G_3'} \alpha$ then $\models_{M4M4} \alpha$.

**Proof.** We use the fact that $G_3'$ has a Hilbert style axiomatization for which a soundness and completeness theorem holds.

Such axiomatic system has Modus Ponens as its unique inference rule. It is not difficult to check that each axiom that defines logic $G_3'$ is a tautology in $M4M4$, hence we can use induction on the length of the proof of formula $\alpha$ in $G_3'$ .

Let $A_1, A_2, ....A_n = \alpha$ be a proof of length $n$ of formula $\alpha$ in $G_3'$. Since Modus Ponens is the only inference rule, there are a couple of indices, say $j, k$ such that $A_j = A_k \to A_n$ where $A_j$ and $A_k$ are theorems in $G_3'$, hence by induction hypothesis they are tautologies in $M4M4$.

According to the truth tables of $M4M4$ if $A_k$ and $A_j = A_k \to A_n$ are theorems then $A_n$ should always take the value 3. therefore $\alpha$ a tautology in $M4M4$.

As a consequence of the previous two results we have that for any given formula $\alpha$, it is a tautology in $M4M4$ if and it is a tautology in $G_3'$ as stated in the next corollary.

**Corollary 2.** For any formula $\alpha$ we have $\models_{G_3'} \alpha$ if and only if $\models_{M4M4} \alpha$.

We state the next result as another corollary.

**Corollary 3.** The formula $\sim\ x$ cannot be expressed in terms of the other connectives and a single atom $x$.

**Proof.** According to Theorem 3 any formula in one atom using exclusively the connectives $\vee, \wedge, \neg$ and $\to$ evaluates to one of the values in $\{0, 2, 3\}$ when the atom takes the value 2, whereas the formula $\sim\ x$ takes the value 1 when the atom $x$ takes the value $2$.

### 3.1 Some Other Properties of Logic $M4M4$

$M4M4$ possesses some other properties that are common in some other logics and are useful to enhance its richness, like for example the De Morgan laws. We establish some of these properties, which are easy to check, in the next result.

**Theorem 5.** The following formulas are tautologies in $M4M4$:

- $\sim\sim \alpha \equiv \alpha$,
- $\sim \alpha \to \neg\alpha$,
- $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$,
- $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$,
- $\sim (\alpha \wedge \beta) \equiv \sim \alpha \vee \sim \beta$,
- $\sim (\alpha \vee \beta) \equiv \sim \alpha \wedge \sim \beta$,
- $\sim \neg\alpha \equiv \neg\neg\alpha$,
- $\sim (\alpha \to \beta) \equiv \backsim\backsim \alpha \wedge \sim \beta$—.

It is worth noting that the reciprocal of the second formula in this theorem is not a tautology. As mentioned before, we provide a form of the substitution theorem.

In order to do that, we need to define a strong equivalence connective since the regular biconditional does not satisfy the substitution property, as shown in the following example.

Last formula in the previous theorem is a tautology, however for the valuation $\alpha = \beta = 2$, the left hand side evaluates to zero and the right hand side evaluates to 1, so when negating both sides of that formula with the strong negation the new formula is not a tautology, since we get 3 on the left hand side and 2 on the right hand side for the same values of the variables:

$$\sim\sim (\alpha \to \beta) \equiv \sim (\backsim\backsim \alpha \wedge \sim \beta). \quad (1)$$

To proceed with our plan we define a new non-primitive connective, which we call the strong equivalence. See Tables 3 and 4:

$$\alpha \Leftrightarrow \beta := (\alpha \leftrightarrow \beta) \wedge (\sim \alpha \Leftrightarrow \sim \beta). \quad (2)$$

**Definition 1.**

$$\varphi\left[\frac{\psi}{\rho}\right] = \begin{cases} \varphi & \text{if } \varphi \text{ is atomic and different from } \rho, \\ \psi & \text{if } \varphi = \rho. \end{cases} \quad (3)$$

**Table 3.** Truth table: standard bi-conditional connective in $M4M4$

| $x$ | $y$ | $x \leftrightarrow y$ |
|---|---|---|
| 0 | 0 | 3 |
| 0 | 1 | 3 |
| 0 | 2 | 0 |
| 0 | 3 | 0 |
| 1 | 0 | 3 |
| 1 | 1 | 3 |
| 1 | 2 | 1 |
| 1 | 3 | 1 |
| 2 | 0 | 0 |
| 2 | 1 | 1 |
| 2 | 2 | 3 |
| 2 | 3 | 2 |
| 3 | 0 | 0 |
| 3 | 1 | 1 |
| 3 | 2 | 2 |
| 3 | 3 | 3 |

**Table 4.** Truth table: strong bi-conditional connective in $M4M4$

| $x$ | $y$ | $x \leftrightarrow y$ | $\sim x \leftrightarrow \sim y$ | $x \Leftrightarrow y$ |
|---|---|---|---|---|
| 0 | 0 | 3 | 3 | 3 |
| 0 | 1 | 3 | 2 | 2 |
| 0 | 2 | 0 | 1 | 0 |
| 0 | 3 | 0 | 0 | 0 |
| 1 | 0 | 3 | 2 | 2 |
| 1 | 1 | 3 | 3 | 3 |
| 1 | 2 | 1 | 1 | 1 |
| 1 | 3 | 1 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 2 | 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | 3 | 3 |
| 2 | 3 | 2 | 3 | 2 |
| 3 | 0 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 |
| 3 | 2 | 2 | 3 | 2 |
| 3 | 3 | 3 | 3 | 3 |

In the case $\varphi$ is not atomic then $\varphi = \varphi_1 \Box \varphi_2$ (where $\Box$ is any of the binary connectives) or $\varphi = \neg\varphi_1$.

For the first case we define:

$$\varphi_1 \Box \varphi_2 \left[\frac{\psi}{\rho}\right] = \varphi_1 \left[\frac{\psi}{\rho}\right] \Box \varphi_2 \left[\frac{\psi}{\rho}\right]. \qquad (4)$$

For the second case we define:

$$\neg(\varphi_1) \left[\frac{\psi}{\rho}\right] = \neg\varphi_1 \left[\frac{\psi}{\rho}\right], \quad \text{or} \quad \sim (\varphi_1) \left[\frac{\psi}{\rho}\right] = \sim \varphi_1 \left[\frac{\psi}{\rho}\right]. \qquad (5)$$

Finally, we present a weak version of the substitution theorem for $M4M4$.

**Theorem 6.**

$$\models \psi_1 \Leftrightarrow \psi_2 \text{ then } \models \varphi \left[\frac{\psi_1}{\rho}\right] \Leftrightarrow \varphi \left[\frac{\psi_2}{\rho}\right]. \qquad (6)$$

**Proof.** The proof is done by induction on the length of $\varphi$.

1. If $\varphi = \rho$ then for each $i$, $\varphi \left[\frac{\psi_i}{\rho}\right] = \varphi$, and the result follows from the induction hypothesis.
2. If $\varphi$ is an atom different from $\rho$ then there is no substitution to be done and the result follows.
3. If $\varphi = \varphi_1 \Box \varphi_2$ then by induction hypothesis

$$\models \varphi_1 \left[\frac{\psi_1}{\rho}\right] \Leftrightarrow \varphi_1 \left[\frac{\psi_2}{\rho}\right], \quad \text{and} \quad \models \varphi_2 \left[\frac{\psi_1}{\rho}\right] \Leftrightarrow \varphi_2 \left[\frac{\psi_2}{\rho}\right]. \qquad (7)$$

According to the strong bi-conditional truth table, we know that any interpretation gives the same truth values to:

$$\varphi_i \left[\frac{\psi_1}{\rho}\right] \text{ and } \varphi_i \left[\frac{\psi_2}{\rho}\right], \qquad (8)$$

And we also know that the truth values of:

$$(\varphi_1 \Box \varphi_2) \left[\frac{\psi_2}{\rho}\right], \qquad (9)$$

Depend on those truth values solely, hence the result follows.

4. If $\varphi = \neg\varphi_1$, then under any interpretation the truth values of:

$$\varphi_1 \left[\frac{\psi_1}{\rho}\right], \quad (10)$$

Are the same as those for:

$$\varphi_1 \left[\frac{\psi_2}{\rho}\right], \quad (11)$$

By hypothesis, therefore as in the previous case, the truth values of:

$$\neg\varphi_1 \left[\frac{\psi_1}{\rho}\right], \quad (12)$$

Are the same as those for:

$$\neg\varphi_1 \left[\frac{\psi_2}{\rho}\right]. \quad (13)$$

Then it follows that:

$$\models \neg\varphi_1 \left[\frac{\psi_1}{\rho}\right] \Leftrightarrow \neg\varphi_1 \left[\frac{\psi_2}{\rho}\right]. \quad (14)$$

5. If $\varphi = \sim \varphi_1$ This case follows exactly as the previous one.

## 4 Conclusions and Future Work

In this paper, we introduce a new 4-valued logic called $M4M4$. It can be used as a formalism to define two logic programming semantics: stable and p-stable.

These logic programming semantics could be used to define the semantics of deductive databases. As future work we propose the study of properties of this logic and the comparison of it to some well known paraconsistent logics.

## References

1. **Affonso da Costa, N. C. (1974).** On the theory of inconsistent formal systems. Notre Dame Journal of Formal Logic, Vol. 15, No. 4. DOI: 10.1305/ndjfl/1093891487.
2. **Galindo, M. O., Navarro-Perez, J. A., Arrazola-Ramirez, J. R., Macias, V. B. (2006).** Logics with common weak completions. Journal of Logic and Computation, Vol. 16, No. 6, pp. 867–890. DOI: 10.1093/logcom/exl013.
3. **Mendelson, E. (1987).** Introduction to Mathematical Logic.
4. **Ortiz, M., Osorio, M. (2007).** Strong negation and equivalence in the safe belief semantics. Journal of Logic and Computation, Vol. 17, No. 3, pp. 499–515. DOI: 10.1093/logcom/exm011.
5. **Osorio Galindo, M., Carballido Carranza, J. L. (2008).** Brief study of $G'_3$ logic. Journal of Applied Non-Classical Logics, Vol. 18, No. 4, pp. 475–499. DOI: 10.3166/jancl.18.475-499.
6. **Solver, A. (2022).** Web location of DLV: www.dlvsystem.com/dlvdb/.

*Article received on 04/04/2022; accepted on 08/06/2022. Corresponding author is Mauricio Osorio.*