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An Algebraic Study of the First Order Version of some Implicational Fragments of Three-Valued Łukasiewicz Logic

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Abstract. In this paper, some implicational fragments of trivalent Łukasiewicz logic are studied and the propositional and first-order logic are presented. The maximal consistent theories are studied as Monteiro's maximal deductive systems of the Lindenbaum-Tarski algebra in both cases. Consequently, the adequacy theorems with respect to the suitable algebraic structures are proven.

Keywords. Trivalent Hilbert algebras, modals operators, 3-valued Gödel logic, first-order logics.

1 Introduction and Preliminaries

In 1923, Hilbert proposed studying the implicative fragment of *classical propositional calculus*. This fragment is well-known as *positive implicative propositional calculus* and its study was started by Hilbert and Bernays in 1934. The following axiom schemas define this calculus:

(E1) $\alpha \to (\beta \to \alpha)$,

(E2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$,

and the inference rule *modus ponens* is:

(MP) $\dfrac{\alpha, \alpha \to \beta}{\beta}$.

In 1950, Henkin introduced the *implicative models* as algebraic models of the positive implicative calculus. Later, A. Monteiro renamed them as *Hilbert algebras* and his Ph. D. student Diego ([8]) made one of the most important contributions to these algebraic structures.

In particular, this author proved that the class of Hilbert algebras is an equational class, that is to say, it is possible to characterize the class via certain equations.

Moreover, Diego proved that the positive implicative propositional calculus is decidable by means of using algebraic technical tools.

On the other hand, Thomas in [26] considered the n-valued positive implicative calculus, with signature $\{\to, 1\}$, as a calculus that has a characteristic matrix $\langle A, \{1\}\rangle$ where $\{1\}$ is the set of designated elements and the algebra $A = (\mathbb{C}_n, \to, 1)$ is defined as follows:

$$\mathbb{C}_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\},$$

and

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & y < x \end{cases}.$$

This author proved that for this calculus we must add the following axiom to the positive implicative calculus:

(E3) $T_n(\alpha_0, \cdots, \alpha_{n-1}) = \beta_{n-2} \to (\beta_{n-3} \to (\cdots \to (\beta_0 \to \alpha_0) \cdots))$, where

$\beta_i = (\alpha_i \to \alpha_{i+1}) \to \alpha_0$ for all $i, 0 \leq i \leq n-2$.

Table 1

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

The algebraic counterpart of n-valued positive implicative calculus was studied in [15] where the axiom (E3) is translated by the equation $T_n = 1$ to the ones of Hilbert algebras. In particular, in the $n = 3$ case, the variety is generated by an algebra that has this set $\mathbb{C}_3 = \{0, \frac{1}{2}, 1\}$ as support and an implication $\to$ defined by the following table 1.

It is clear that 3-valued Hilbert algebras are Hilbert algebras that verify the following identity:

(IT3) $((x \to y) \to z) \to (((z \to x) \to z) \to z) = 1.$

It is important to note that the implication defined in Table 1 characterizes the implication of 3-valued Gödel logic that we call G3.

Paraconsistent extensions of 3-valued Gödel logic were studied as a tool for knowledge representation and nonmonotonic reasoning, [21, 20]. Particularly, Osorio and his collaborators showed that some of these logics can be used to express interesting nonmonotonic semantics. In addition, these paraconsistent systems were also studied under a mathematical logic point of view as we can see in the following papers: [22, 12, 17, 19, 18]. To see other applications of three-valued logic to other fields the reader can consult [5].

In this paper, we will study implicative fragments of G3 enriched with certain modal operators that we call Moisil's operators. In this setting, recall that Moisil introduced 3-valued Łukasiewicz algebras (or 3-valued Łukasiewicz-Moisil algebras) as algebraic models of 3-valued logic proposed by Łukasiewicz. It is well-known, and part of folklore, that the class of 3-valued Łukasiewicz algebras is term equivalent to the one of 3-valued MV-algebras (see, for instance, [2]). Recall that an algebra $(A, \wedge, \vee, \sim, \nabla, 0, 1)$ is a 3-valued Łukasiewicz algebra if the following conditions hold: (L0) $x \vee 1 = 1$, (L1) $x \wedge (x \vee y) = x$, (L2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$, (L3) $\sim\sim x = x$, (L4) $\sim (x \wedge y) = \sim x \vee \sim y$, (L5) $\sim x \vee \nabla x = 1$, (L6) $\sim x \wedge x = \sim x \wedge \nabla x$, and (L7) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$. It is well known that each 3-valued Łukasiewicz algebra is a De Morgan algebra because equations (L0) to (L4) hold, [2, Definition 2.6]. In general, to see more technical aspects of Łukasiewicz-Moisil algebras, the reader can consult [2].

On the other hand, the characteristic matrix of logic from trivalent Łukasiewicz algebras has the operators $\wedge$, $\vee$, $\sim$, ∇ (possibility operator) and $\triangle$ (necessity operator) over the chain $\mathbb{C}_3 = \{0, \frac{1}{2}, 1\}$, and they are defined by the next table:

Table 2

x	$\sim x$	∇x	$\triangle x$
0	1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	1	0
1	0	1	1

In addition, the implication $\to$ defined in Table 1 can be obtained from the operators $\wedge$, $\vee$, $\sim$, ∇ and $\triangle$ by the following formula:

$$x \to y = \triangle \sim x \vee y \vee (\nabla \sim x \wedge \nabla y).$$

Moreover, it is not hard to see that $\nabla x = (x \to \triangle x) \to \triangle x$. In this setting, the algebraic structures in the signature $\{\to, \triangle\}$ were defined and studied by Canals-Frau and Figallo in [6, 7]; these structures can be seen as certain $\{\to, \triangle\}$-fragments of 3-valued Łukasiewicz algebras.

The rest of the paper is organized as follows: in section 2, we introduce and study the class of modal 3-valued Hilbert algebras with supremum and also, as an application of our algebraic work, we present a Hilbert calculus for the fragment with disjunction soundness and completeness, in a strong version, with respect to this class of algebras. In Section 3, we study the first-order logic for the fragment with disjunction by means of an adaptation of the Rasiowa's technique ([25]) using our algebraic work for the propositional case. In the last Section, we discuss the possibility to applied our proofs to other classes of algebras.

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2 Trivalent Modal Hilbert Algebras With Supremum

In this section, we will introduce and study algebraically the trivalent modal Hilbert algebra with supremum that we denote $H_3^{\vee,\triangle}$-algebras. From this algebraic work, we present a sound and complete calculus w.r.t. the class of $H_3^{\vee,\triangle}$-algebras in propositional case.

For the sake of brevity, we only introduce those essential notions of *Hilbert algebras* that we need, thought not in full detail. Anyway, for more information about these algebras, the reader can consult the bibliography.

Now, recall that a Hilbert algebra is an algebra $(A, \to, 1)$ such that for all $x, y, z \in A$ verifies:

(H1) $x \to (y \to x) = 1$,

(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

(H3) if $x \to y = 1$, $y \to x = 1$, then $x = y$.

Furthermore, we say $(A, \to, 1)$ is a 3-valued Hilbert algebra if verifies the following equation: (IT3) $((x \to y) \to z) \to (((z \to x) \to z) \to z) = 1$.

The following lemma is well-known and the proof can be found in [8].

Lemma 2.1 *Let A be a Hilbert algebra. The following properties are satisfied for every $x, y, z \in A$:*

(H4) *if $x = 1$ and $x \to y = 1$, then $y = 1$;*

(H5) *the relation $\leq$ defined by $x \leq y$ iff $x \to y = 1$, which is an order relation on A and 1 is the last element;*

(H6) $x \to x = 1$;

(H7) $x \leq y \to x$;

(H8) $x \to (y \to z) \leq (x \to y) \to (x \to z)$;

(H9) $x \to 1 = 1$;

(H10) $x \leq y$ *implies* $z \to x \leq z \to y$;

(H11) $x \leq y \to z$ *implies* $y \leq x \to z$;

(H12) $x \to ((x \to y) \to y) = 1$,

(H13) $1 \to x = x$;

(H14) $x \leq y$ *implies* $y \to z \leq x \to z$,

(H15) $x \to (y \to z) = y \to (x \to z)$;

(H16) $x \to (x \to y) = x \to y$;

(H17) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$;

(H18) $x \to (y \to z) = (x \to y) \to (x \to z)$;

(H19) $((x \to y) \to y) \to y = x \to y$.

In the following, we present a definition of the equational class of 3-valued modal Hilbert algebra that was introduced in [6].

Definition 2.2 *An algebra $(A, \to, \triangle, 1)$ is said to be a 3-valued modal Hilbert algebra if its reduct $(A, \to, 1)$ is a 3-valued Hilbert algebra and $\triangle$ verifies the following identities:*

$(\mathrm{M1})$ $\triangle x \to x = 1$,

$(\mathrm{M2})$ $((y \to \triangle y) \to (x \to \triangle\triangle x)) \to \triangle(x \to y) = \triangle x \to \triangle\triangle y$, *and*

$(\mathrm{M3})$ $(\triangle x \to \triangle y) \to \triangle x = \triangle x$.

Moreover, we define a new connective by $\nabla x = (x \to \triangle x) \to \triangle x$.

Now, consider the following Definition that we introduce for the first time.

Definition 2.3 *An algebra $\mathbf{A} = \langle A, \to, \vee, \triangle, 1\rangle$ is said to be a trivalent modal Hilbert algebra with supremum if the following properties hold:*

(1) *the reduct $\langle A, \vee, 1\rangle$ is a join-semilattice with greatest element 1, and the conditions (a) $x \to (x \vee y) = 1$ and (b) $(x \to y) \to ((x \vee y) \to y) = 1$ hold. Besides, given $x, y \in A$ such that there exists the infimum of $\{x, y\}$, denoted by $x \wedge y$, then $\triangle(x \wedge y) = \triangle x \wedge \triangle y$.*

(2) *The reduct $\langle A, \to, \triangle, 1\rangle$ is a $\triangle H_3$-algebra.*

From now on, we denote with $\mathbf{A}$ the $H_3^{\vee,\triangle}$-algebra $\langle A, \to, \vee, \triangle, 1\rangle$ and with A its support. Next, we will show some properties that will be very useful for the rest of this section.

Let us notice that there is an $H_3^{\vee,\triangle}$-algebra A in which the infimum can not be defined. To see that, take some subalgebras of $\mathbb{C}_3^{\to,\vee} \times \mathbb{C}_3^{\to,\vee}$ where $\times$ is the direct product.

The fragment with infimum has been studied in [24] that will comment in the following Remark.

Remark 2.4 *In [24], the class of 3-valued modal Hilbert algebra with imfimum ($i\triangle H_3$-algebra) was defined as follows: An algebra $\langle A, \to, \wedge, \triangle, 1\rangle$ is said to be an $i\triangle H_3$-algebra if the following conditions hold: (1) the reduct $\langle A, \to, \triangle, 1\rangle$ is a 3-valued modal Hilbert algebra; (2) the following identities hold: (iH_1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, (iH_2) $x \wedge x = x$, (iH_3) $x \wedge (x \to y) = x \wedge y$, and (iH_4) $(x \to (y \wedge z)) \to ((x \to z) \wedge (x \to y)) = 1$.*

Let us observe that for each $i\triangle H_3$-algebra $\mathbf{A}$ and for every $x, y \in A$, we can define the supremum of $\{x, y\}$ in the following way:

$$x \vee y \overset{def}{=} ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Indeed, let $a, b \in A$ and put $c = ((a \to b) \to b) \wedge ((b \to a) \to a)$. Since $x \leq (x \to y) \to y$ and $x \leq (y \to x) \to x$ hold and there exists the infimum $((x \to y) \to y) \wedge ((y \to x) \to x)$, then c is upper bound of the set $\{a, b\}$. Now, let us suppose that d is another upper bound of $\{a, b\}$ such that $c \not\leq d$. Thus, there exists an irreducible deductive system P such that $c \in P$ and $d \notin P$ [8, Corolario 1]. Besides, since $a, b \leq d$ then $a, b \notin P$. On the other hand, as A is a trivalent Hilbert algebra and according to [14, Théorème 4.1], we have $a \to b \in P$ or $b \to a \in P$. Now, if we suppose that $a \to b \in P$ and since $c \leq (b \to a) \to a$, then we can infer that $a \in P$, which is a contradiction. If we consider the case $b \to a \in P$, we also obtain a contradiction. Thus, c is the supremum of $\{a, b\}$. Therefore, each $i\triangle H_3$-algebra is a relatively pseudocomplemented lattice *since $x \wedge z \leq y$ iff $x \leq z \to y$, see [25]. From the latter, we have that each $i\triangle H_3$-algebra is a distributive lattice. It is possible to see that every finite and complete $i\triangle H_3$-algebra is a 3-valued Łukasiewicz algebra. To see the details, the reader can consult Section 3 of [24].*

Lemma 2.5 *For a given $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$ and $x, y, z \in A$, then the following properties hold:*

(1) $\triangle 1 = 1$;

(2) $\triangle(x \to y) \to (\triangle x \to \triangle y) = 1$;

(3) *if* $x \to y = 1$, *then* $x \vee y = y$;

(4) *if* $x \to z = 1$ *and* $y \to z = 1$, *then* $(z \vee y) \to z = 1$;

(5) $x \to (x \vee y) = 1$,

(6) $(x \to z) \to ((y \to z) \to ((x \vee y) \to z)) = 1$;

(7) $\triangle(x \vee y) = \triangle x \vee \triangle y$;

(8) $\nabla(x \vee y) = \nabla x \vee \nabla y$.

Proof. It is routine. $\square$

Definition 2.6 *For a given $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$ and $D \subseteq A$. Then, D is said to be a deductive system if (D1) $1 \in D$, and (D2) if $x, x \to y \in D$ imply $y \in D$. Additionally, we say that D is a modal if: (D3) $x \in D$ implies $\triangle x \in D$. Moreover, D is said to be maximal if for every modal deductive system M such that $D \subseteq M$ implies $M = A$ or $M = D$.*

Given a $H_3^{\vee,\triangle}$-algebra-algebra $\mathbf{A}$ and $\{H_i\}_{i\in I}$ a family of modal deductive systems of A, then it is easy to see that $\bigcap_{i\in I} H_i$ is a modal deductive system. Thus, we can consider the notion of modal deductive system generated by H, denoted $[H)_m$, as an intersection of all modal deductive system D such that $D \subseteq H$. The deductive system generated by H, denoted $[H)$, verify that $[H) = \{x \in A : \text{there exist } h_1, \cdots, h_k \in H \text{ such as } h_1 \to (h_2 \to \cdots \to (h_k \to x)\cdots) = 1\}$ where k is a finite integer, see [8]. Now, we will introduce the following notation:

$$(x_1, \ldots, x_{n-1}; x_n) = \begin{cases} x_n, & if\, n = 1, \\ x_1 \to (x_2, \ldots, x_{n-1}; x_n), & if\, n > 1. \end{cases}$$

Hence, we can write:

$$[H) = \{x \in A : \text{there exist } h_1, \ldots, h_k \in D_1 : (h_1, \ldots, h_k; x) = 1\}.$$

Then, we have the following result:

Proposition 2.7 *Let $\mathbf{A}$ be a $H_3^{\vee,\triangle}$-algebra, suppose that $H \subseteq A$ and $a \in A$. Then the following properties hold:*

(i) $[H)_m = \{x \in A : \text{there exist } h_1, \cdots, h_k \in H : (\triangle h_1, \ldots, \triangle h_k; x) = 1\}$;

(ii) $[a)_m = [\triangle a)$, *where* $[b)$ *is the set* $[\{b\})$;

(iii) $[H \cup \{a\})_m = \{x \in A : \triangle a \to x \in [H)_m\}$.

Proof. It is routine. □

Lemma 2.8 *Given a $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$, there exists a lattice-isomorphism between the poset of congruences of A and the poset of the modal deductive systems of A.*

Proof. It is well-known that the set of congruences of Hilbert algebra A is lattice-isomorphic to the set of all deductive systems. For each deductive system D we have the relation $R(D) = \{(x, y) : x \to y, y \to x \in D\}$ which is a congruence of A, such that the class of 1 verifies $|1|_{R(D)} = D$. In addition, for each congruence θ of A, the class of $|1|_\theta$ is a deductive system and $R(|1|_\theta) = \theta$. From the latter and Lemma 2.5 (1) and (2), we can infer that every congruence θ for a given A respect $\triangle$ and $|1|_\theta$ is a modal deductive system. □

For each $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$, we can define a new binary operation $\rightarrowtail$ named weak implication such that: $x \rightarrowtail y = \triangle x \to y$.

Lemma 2.9 *Let $\mathbf{A}$ be a $H_3^{\vee,\triangle}$-algebra, for any $x, y, z \in A$ the following properties hold:*

(wi1) $1 \rightarrowtail x = x$;

(wi2) $x \rightarrowtail x = 1$;

(wi3) $x \rightarrowtail \triangle x = 1$;

(wi4) $x \rightarrowtail (y \rightarrowtail z) = (x \rightarrowtail y) \rightarrowtail (x \rightarrowtail z)$;

(wi5) $x \rightarrowtail (y \rightarrowtail x) = 1$;

(wi6) $((x \rightarrowtail y) \rightarrowtail x) \rightarrowtail x = 1$.

Proof. The proof immediately follows from the very definitions; and, it can be consulted [24, Lema 2.4.2]. □

Let $\mathbf{A}$ an $H_3^{\vee,\triangle}$-algebra and suppose a subset $D \subseteq A$, we say that D is a weak deductive system (w.d.s.) if $1 \in D$, and $x, x \rightarrowtail y \in D$ imply $y \in D$. It is not hard to see that the set of modal deductive systems is equal to the set of weak deductive systems. We denote by $\mathcal{D}_w(A)$ the set of weak deductive systems of a Hilbert algebra.

Now, for a given $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$ and a (weak) deductive system D of A, D is said to be a maximal if for every (weak) deductive system M such that $D \subseteq M$, then $M = A$ or $M = D$. Besides, let us consider the set of all maximal w.d.s. $\mathcal{E}_w(A)$. A. Monteiro gave the following definition in order to characterize maximal deductive systems:

Definition 2.10 *(A. Monteiro) Let $\mathbf{A}$ be a $H_3^{\vee,\triangle}$-algebra, $D \in \mathcal{D}_w(A)$ and $p \in A$. We say that D is a weak deductive system tied to p if $p \notin D$ and for any $D' \in \mathcal{D}_w(A)$ such that $D \subsetneq D'$, then $p \in D'$.*

The importance for introducing the notion of weak deductive systems is to prove that every maximal weak deductive system is a weak deductive system tied to some element of a given $H_3^{\vee,\triangle}$-algebra, A. Conversely, and using (wi6), we can prove every w.d.s is a maximal weak deductive systems. Moreover, from (wi4), (wi5) and (wi1) and using A. Monteiro's techniques, we also can prove that $\{1\} = \bigcap_{M \in \mathcal{E}_w(A)} M$. To see the details of the proof, see Sections 2.4, 2.5 and 2.6 of [24].

In what follows, we will consider the quotient algebra $\mathbf{A}/M$ defined by $a \equiv_M b$ iff $a \to b, b \to a \in M$ and the canonical projection $q_M : \mathbf{A} \to \mathbf{A}/M$ defined by $q_M = |x|_M$ where $|x|_M$ denotes the equivalence class of x generated by M.

Lemma 2.11 *Let $\mathbf{A}$ be a $H_3^{\vee,\triangle}$-algebra. Then, the map $\Phi : \mathbf{A} \longrightarrow \prod_{M \in \mathcal{E}_w(A)} A/M$ defined by $\Phi(x)(M) = q_M(x)$ is a homomorphism; that is to say, the variety of $H_3^{\vee,\triangle}$-algebras is semisimple.*

Proof. Taking $\prod_{\alpha\in\mathcal{E}_w(A)} A/M_\alpha = \{f : \mathcal{A} \to \bigcup_{\alpha\in\mathcal{E}_w(A)} A/M_\alpha : f(\alpha) \in A/M_\alpha$ for every $\alpha \in \mathcal{E}_w(A)\}$ and $\mathcal{E}_w(A)$ is the set of maximal w.d.s. defined before. Let us define $\Phi : A \to \prod_{\alpha\in\mathcal{E}_w(A)} A/M_\alpha$ such that for every α we have that $\Phi(\alpha) = f_a$ where $f_a(\alpha) = q_\alpha(a) = |a|_\alpha \in A/M_\alpha$ with $a \in A$. It is not hard to see that Φ is a homomorphism in view of the fact that $\equiv_{M_\alpha}$ is a congruence relation. Now, from the fact that $\{1\} = \bigcap_{M\in\mathcal{E}_w(A)} M$, it is possible to see that Φ is one-to-one function which completes the proof. □

The construction of the following homomorphism is fundamental to obtaining the generating algebras of the variety of $H_3^{\vee,\triangle}$-algebra. Moreover, this homomorphism will play a central *role* in the adequacy theorems in a propositional and first-order version of logic, as we will see later on.

In the next, we consider the algebras $\mathbb{C}_3^{\to,\vee} = \langle\{0, \frac{1}{2}, 1\}, \to, \vee, \triangle, 1\rangle$ and $\mathbb{C}_2^{\to,\vee} = \langle\{0,1\}, \to, \vee, \triangle, 1\rangle$. We denote $\mathbb{C}_3$ and $\mathbb{C}_2$ the support of $\mathbb{C}_3^{\to,\vee}$ and $\mathbb{C}_2^{\to,\vee}$, respectively; besides, the operation $\vee$ is the maximum on the corresponding chain.

Theorem 2.12 *Let M be a non-trivial maximal modal deductive system of an $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$. Let us consider the sets $M_0 = \{x \in A : \nabla x \notin M\}$ and $M_{1/2} = \{x \in A : x \notin M, \nabla x \in M\}$, and the map $h : A \longrightarrow \mathbb{C}_3$ defined by*

$$h(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ 1/2 & \text{if } x \in M_{1/2} \\ 1 & \text{if} x \in M. \end{cases}$$

Then, h is a homomorphism from $\mathbf{A}$ into $\mathbb{C}_3^{\to,\vee}$ such that $h^{-1}(\{1\}) = M$.

Proof. We shall prove only that $h(x \vee y) = h(x) \vee h(y)$, for the rest of the proof can be done in a similar manner.

(1) Let $x \in M$ and $y \in A$. Taking into account (5) of Lemma 2.5, we have that $x \to (x \vee y) = 1$. Thus, from D_1) and D_2) then $x \vee y \in M$.

(3) Let us consider $x, y \in M_0$ and suppose that $\nabla(x \vee y) \in M$, then by (8) of Lemma 2.5, we have that $\nabla x \vee \nabla y \in M$. Thus, according to (6) of Lemma 2.5, we infer that $(\nabla x \to \nabla x) \to ((\nabla y \to \nabla x) \to ((\nabla x \vee \nabla y) \to \nabla x)) = 1$. So, from D_1), D_2) and (H6), we can obtain that $(\nabla y \to \nabla x) \to ((\nabla x \vee \nabla y) \to \nabla x) \in M$. Since $\nabla x \notin M$, we can infer that $\triangle\nabla y \to \nabla x \in M$ and so, we have $\nabla y \to \nabla x \in M$. Form the latter and D_2), we can write $(\nabla x \vee \nabla y) \to \nabla x \in M$. Therefore, $\nabla x \in M$ which is impossible, then $\nabla(x \vee y) \notin M$.

(4) If $x \in M_0$ and $y \in M_{1/2}$, since $\nabla y \to (\nabla x \vee \nabla y) = 1$ and $\nabla y \in M$, we can infer that $\nabla x \vee \nabla y \in M$. Now, let us suppose that $x \vee y \in M$. From (6) of Lemma 2.5, we can write $(x \to y) \to ((y \to y) \to ((x \vee y) \to y)) = 1$. Thus, $x \to y \in M$ and then, $y \in M$ which is a contradiction. Therefore, $x \vee y \in M_{1/2}$.

(5) If $x \in M_{1/2}$ and $y \in M_0$, we can prove that $x \vee y \in M_{1/2}$ in a similar way to (4).

(6) Suppose that $x \in M_{1/2}$ and $y \in M_{1/2}$, then from (8) of Lemma 2.5, we have that $\nabla(x \vee y) \in M$. On the other hand, let us suppose $x \vee y \in M$, thus by (6) of Lemma 2.5, we infer that $(x \to x) \to ((x \to y) \to ((x \vee y) \to x)) = 1$. Hence, since $x \to y \in M$, we can write $x \in M$ which is a contradiction. Therefore, $x \vee y \in M_{1/2}$.

□

According to Lemma 2.11 and Theorem 2.12, and well-known facts about universal algebra, we have proved the following Corollary.

Corollary 2.13 *The variety of $H_3^{\vee,\triangle}$-algebras is semisimple. Moreover, the algebras:*

$$\mathbb{C}_3^{\to,\vee} = \langle\{0, \frac{1}{2}, 1\}, \to, \vee, \triangle, 1\rangle,$$

and

$$\mathbb{C}_2^{\to,\vee} = \langle\{0, 1\}, \to, \vee, \triangle, 1\rangle.$$

are the unique simple algebras.

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2.1 Propositional Calculus for $H_3^{\vee,\triangle}$-Algebras

Let $\mathfrak{Fm}_s = \langle Fm, \vee, \rightarrow, \triangle\rangle$ be the absolutely free algebra over $\Sigma = \{\rightarrow, \vee, \triangle\}$ generated by a set $Var = \{p_1, p_2, \cdots\}$ of numerable variables. As usual, we say that $\mathfrak{Fm}_s$ is a language over Var and Σ. Consider now the following logic:

Definition 2.14 *We denote by $\mathcal{H}^3_{\vee,\triangle}$ the Hilbert calculus determined by the following axioms and inference rules, where $\alpha, \beta, \gamma, ... \in Fm$:*

Axiom schemas

(Ax1) $\alpha \rightarrow (\beta \rightarrow \alpha)$,

(Ax2) $(\alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$,

(Ax3) $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (((\gamma \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma)$,

(Ax4) $\alpha \rightarrow (\alpha \vee \beta)$,

(Ax5) $\beta \rightarrow (\alpha \vee \beta)$,

(Ax6) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$,

(Ax7) $\triangle\alpha \rightarrow \alpha$,

(Ax8) $\triangle(\triangle\alpha \rightarrow \beta) \rightarrow (\triangle\alpha \rightarrow \triangle\beta)$,

(Ax9) $((\beta \rightarrow \triangle\beta) \rightarrow (\alpha \rightarrow \triangle(\alpha \rightarrow \beta))) \rightarrow \triangle(\alpha \rightarrow \beta)$,

(Ax10) $((\triangle\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow ((\triangle\alpha \rightarrow \gamma) \rightarrow \gamma)$.

Inference Rules

(MP) $\dfrac{\alpha, \alpha \rightarrow \beta}{\beta}$, (NEC) $\dfrac{\alpha}{\triangle\alpha}$.

Assume that $\nabla\alpha := (\alpha \rightarrow \triangle\alpha) \rightarrow \triangle\alpha$.

Let $\Gamma \cup \{\alpha\}$ be a set formulas of $\mathcal{H}^3_{\vee,\triangle}$, we define the derivation of α from Γ in usual a way and denote it by $\Gamma \vdash_\vee \alpha$.

Lemma 2.15 *The following rules are derivable in $\mathcal{H}^3_{\vee,\triangle}$:*

(P_s1) $\vdash_\vee (x \vee y) \rightarrow (y \vee x)$;

(P_s2) $\{x \rightarrow y\} \vdash_\vee (x \vee z) \rightarrow (y \vee z)$;

(P_s3) $\{x \rightarrow y, u \rightarrow v\} \vdash_\vee (x \vee u) \rightarrow (y \vee v)$;

$(R_\vee 3)$ $\dfrac{\alpha \rightarrow \beta}{(\alpha \vee \beta) \rightarrow \beta}$.

Proof. It is routine. □

Now, we denote by $\alpha \equiv_\vee \beta$ if conditions $\vdash_\vee \alpha \rightarrow \beta$ and $\vdash_\vee \beta \rightarrow \alpha$ hold. Then,

Lemma 2.16 *$\equiv_\vee$ is a congruence on $\mathfrak{Fm}_s$.*

Proof. We only have to prove that if $\alpha \equiv_\vee \beta$ and $\gamma \equiv_\vee \delta$, then $\alpha \vee \gamma \equiv_\vee \beta \vee \delta$, which follows immediately from (P_s3). □

Since the $\equiv_\vee$ is a congruence, it allows us to define the quotient algebra $\mathfrak{Fm}_s/\equiv_\vee$ that is so-called the Lindenbaum-Tarski algebra.

Theorem 2.17 *The algebra $\mathfrak{Fm}_s/\equiv_\vee$ is a $H_3^{\vee,\triangle}$-algebra by defining: $|\alpha| \rightarrow |\beta| = |\alpha \rightarrow \beta|$, $|\alpha| \vee |\beta| = |\alpha \vee \beta|$ and $1 = |\beta \rightarrow \beta| = \{\alpha \in \mathfrak{Fm}_s : \vdash_\vee \alpha\}$, where $|\delta|$ denotes the equivalence class of the formula δ.*

Proof. We only have to prove $\mathfrak{Fm}_s/\equiv_\vee$ is a join-semilattice and the axioms (a) and (b) from Definition 2.3 (2). So, the first part follows from (Ax4), (Ax5) and (Ax6), and the second one follows from axioms (Ax4) and $(R_\vee 3)$. □

Now, we will introduce some useful notions in order to prove a strong version of Completeness Theorem for $\mathcal{H}^3_{\vee,\triangle}$ w.r.t. the class of $H_3^{\vee,\triangle}$-algebras.

Recall that a logic defined over a signature $\mathcal{S}$ is a system $\mathcal{L} = \langle For, \vdash\rangle$ where For is the set of formulas over $\mathcal{S}$ and the relation $\vdash \subseteq \mathcal{P}(For) \times For$, $\mathcal{P}(A)$ is the set of all subsets of A. The logic $\mathcal{L}$ is said to be a Tarskian if it satisfies the following properties, for every set $\Gamma \cup \Omega \cup \{\varphi, \beta\}$ of formulas:

(1) if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$,

(2) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Omega$, then $\Omega \vdash \alpha$,

(3) if $\Omega \vdash \alpha$ and $\Gamma \vdash \beta$ for every $\beta \in \Omega$, then $\Gamma \vdash \alpha$.

A logic $\mathcal{L}$ is said to be finitary if it satisfies the following:

(4) if $\Gamma \vdash \alpha$, then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash \alpha$.

Definition 2.18 *Let $\mathcal{L}$ be a Tarskian logic and let $\Gamma \cup \{\varphi\}$ be a set of formulas, we say that Γ is a theory. In addition, Γ is said to be a consistent theory if there is φ such that $\Gamma \nvdash_{\mathcal{L}} \varphi$. Furthermore, we say that Γ is a maximal consistent theory if $\Gamma, \psi \vdash_{\mathcal{L}} \varphi$ for any $\psi \notin \Gamma$; and, in this case, we also say Γ non-trivial maximal respect to φ.*

A set of formulas Γ is closed in $\mathcal{L}$ if the following property holds for every formula φ: $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\varphi \in \Gamma$. It is easy to see that any maximal consistent theory is a closed one.

Lemma 2.19 (Lindenbaum-Łos) *Let $\mathcal{L}$ be a Tarskian and finitary logic. Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that $\Gamma \nvdash \varphi$. Then, there exists a set of formulas Ω such that $\Gamma \subseteq \Omega$ with Ω maximal non-trivial with respect to φ in $\mathcal{L}$.*

Proof. It can be found [27, Theorem 2.22]. $\square$

It is worth mentioning that, by the very definitions, $\mathcal{H}^3_{\vee,\triangle}$ is a Tarskian and finitary logic and then, we have the following:

Theorem 2.20 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. Let $\Gamma/\equiv_\vee = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the trivalent modal Hilbert algebra with supremum $\mathfrak{Fm}_s/\equiv_\vee$, then:*

1. *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$ then $\beta \in \Gamma$,*
2. *$\Gamma/\equiv_\vee$ is a modal deductive system of $\mathfrak{Fm}/\equiv_\vee$. Also, if $\overline{\varphi} \notin \Gamma/\equiv_\vee$ and for any modal deductive system $\overline{D}$ which contains properly to $\Gamma/\equiv_\vee$, then $\overline{\varphi} \in \overline{D}$.*

Proof. Taking into account $\alpha \in \Gamma$ and $\alpha \equiv_\vee \beta$, we have that $\vdash \alpha \to \beta$ and $\vdash \beta \to \alpha$. Therefore, $\beta \in \Gamma$. Besides, it is not hard to see that D_1), D_2) and D_3) are valid, see Definition 2.6.

On the other hand, let $\overline{D}$ be mds that contains $\Gamma/\equiv_\vee$ and so, there is $\overline{\gamma} \in \overline{D}$ such that $\overline{\gamma} \notin \Gamma/\equiv_\vee$. Now, we have that $\gamma \notin \Gamma$ and therefore, $\Gamma \cup \{\gamma\} \vdash \varphi$. From the latter and taking into account $D = \{\alpha : \overline{\alpha} \in \overline{D}\}$, we can infer that $D \vdash \varphi$. Now, let us suppose that $\alpha_1, ..., \alpha_n$ is a derivation from D. We shall prove by induction over the length of the derivation that $\overline{\alpha_n} \in \overline{D}$. Indeed:

If $n = 1$, then α_1 is an instance of an axiom or otherwise $\alpha_1 \in D$. From the first case, we have $\vdash \alpha_1$ and then $\Gamma \vdash \alpha_1$ which is a contradiction. Then, it only can occur that $\alpha_1 \in D$ which implies $\overline{\varphi} \in \overline{D}$.

Suppose that $\overline{\alpha_k} \in \overline{D}$ if k is less than n. Then, we have the following cases:

1. If φ be the instance of an axiom, then $\Gamma \vdash \varphi$ which is a contradiction. This case can not occur.
2. If $\varphi \in D$, then $\overline{\varphi} \in \overline{D}$.
3. If there exists $\{j, t_1, ..., t_m\} \subseteq \{1, ..., k-1\}$ such that $\alpha_{t_1}, ..., \alpha_{t_m}$ is a derivation of $\alpha_j \to \varphi$, then we have $\overline{\alpha_j \to \varphi} \in \overline{D}$ by induction hypothesis. So, $\overline{\alpha_j} \to \overline{\varphi} \in \overline{D}$. From the latter and since $j < k$, we have $\overline{\alpha_j} \in \overline{D}$ and therefore, $\overline{\varphi} \in \overline{D}$.
4. If there exists $\{j, t_1, ..., t_m\} \subseteq \{1, ..., k-1\}$ such that $\alpha_{t_1}, ..., \alpha_{t_m}$ is a derivation of α_j and suppose that α_n is $\triangle\alpha_j$, then $\overline{\alpha_j} \in \overline{D}$. Now, since $\overline{D}$ is a mds, we have that $\triangle\overline{\alpha_j} \in \overline{D}$. Thus, $\overline{\varphi} \in \overline{D}$, which completes the proof. $\square$

The notion of deductive systems considered in the last Theorem, part 2, was named *Systèmes deductifs liés à "a"* by A. Monteiro, where a is an element of some given algebra such that the congruences are determined by deductive systems [13, pag. 19], see also Definition 2.10.

Recall that for a given $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$, a *logical matrix* for $\mathcal{H}^3_{\vee,\triangle}$ is a pair $\langle A, \{1\}\rangle$ where $\{1\}$ is the set of designated elements. In addition, we say that a homomorphism $v : \mathfrak{Fm}_s \to A$ is a valuation. Then, we say that φ is a *semantical consequence* of Γ and we denote by $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$, if for every $H_3^{\vee,\triangle}$-algebra $\mathbf{A}$ and every valuation v, if $v(\Gamma) = \{1\}$ then $v(\varphi) = 1$.

Corollary 2.21 *Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. Then, there exists a valuation $v : \mathfrak{Fm}_s \to \mathbb{C}_3^{\to,\vee}$ such that $v(\varphi) = 1$ iff $\varphi \in \Gamma$.*

Proof. Taking into account Theorem 2.20, we known that $\Gamma/\equiv_\vee$ is a maximal modal deductive system of $\mathfrak{Fm}_s/\equiv_\vee$. Then, by Theorem 2.12, there is a homomorphism $h : \mathfrak{Fm}_s/\equiv_\vee \to \mathbb{C}_3^{\to,\vee}$ (see Corollary 2.13) such that $h^{-1}(\{1\}) = \Gamma/\equiv_\vee$. Now, consider the canonical projection $\pi : \mathfrak{Fm}_s \to \mathfrak{Fm}_s/\equiv$ defined by $\pi(\alpha) = |\alpha|$, see Theorem 2.17. Now, it is enough to take $v = h \circ \pi$ to end the proof. $\square$

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Theorem 2.22 *(Soundness and Completeness of $\mathcal{H}^3_{\vee,\triangle}$ w.r.t. the class of $H_3^{\vee,\triangle}$-algebras) Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, $\Gamma \vdash_\vee \varphi$ if and only if $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$.*

Proof. Soundness: It is not hard to see that every axiom is valid for every $H_3^{\vee,\triangle}$-algebra A. In addition, satisfaction is preserved by the inference rules.

Completeness: Suppose $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$ and $\Gamma \nvdash_\vee \varphi$. Then, according to Lemma 2.19, there is maximal consistent theory M such that $\Gamma \subseteq M$ and $M \nvdash_\vee \varphi$. From the latter and Corollary 2.21, there is a valuation $\mu : \mathfrak{Fm}_s \to \mathbb{C}_3^{\to,\vee}$ such that $\mu(\Delta) = \{1\}$ but $\mu(\varphi) \neq 1$. □

3 Model Theory and First Order version of the logic of $\mathcal{H}_3^{\vee,\triangle}$ Without Identities

In this section, we will define the first order logic of $\mathcal{H}_3^{\vee,\triangle}$. First, let $\Sigma = \{\to, \vee, \triangle\}$ be the propositional signature of $\mathcal{H}_3^{\vee,\triangle}$, the symbols $\forall$ (universal quantifier) and $\exists$ (existential quantifier), with the punctuation marks (commas and parentheses). Let $Var = \{v_1, v_2, ...\}$ be a numerable set of individual variables. A first order signature Θ is composed of the following elements:

— a set $\mathcal{C}$ of individual constants,

— for each $n \geq 1$, $\mathcal{F}$ a set of functions of arity n,

— for each $n \geq 1$, $\mathcal{P}$ a set of predicates of arity n.

The notions of bound and free variables inside a formula, closed terms, closed formulas (or sentences), and of the term free for a variable in a formula are defined as usual, see [23]. We will denote by T_Θ and $\mathfrak{Fm}_\Theta$ the sets of all terms and formulas, respectively. Given a formula φ, the formula obtained from φ by substituting every free occurrence of a variable x by a term t will be denoted by $\varphi(x/t)$.

Definition 3.1 *Let Θ be a first order signature. The logic $\mathcal{QH}_3^{\vee,\triangle}$ over Θ is defined by Hilbert calculus obtained by extending $\mathcal{H}_3^{\vee,\triangle}$ expressed in the language $\mathfrak{Fm}_\Theta$ by adding the following:*

Axioms Schemas

(Ax11) $\varphi(x/t) \to \exists x\varphi$, *if t is a term free for x in φ,*

(Ax12) $\forall x\varphi \to \varphi(x/t)$, *if t is a term free for x in φ,*

(Ax13) $\triangle\exists x\varphi \leftrightarrow \exists x\triangle\varphi$,

(Ax14) $\triangle\forall x\varphi \leftrightarrow \forall x\triangle\varphi$,

Inferences Rules

(R3) $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$ *where x does not occur free in ψ,*

(R4) $\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$ *where x does not occur free in φ.*

We denote by $\vdash \alpha$ the derivation of a formula α in $\mathcal{QH}_3^{\vee,\triangle}$ and with $\Gamma \vdash \alpha$ the derivation of α from a set of premises Γ. These notions are defined as the usual way. Furthermore, we denote $\vdash \varphi \leftrightarrow \psi$ as an abbreviation of $\vdash \varphi \to \psi$ and $\vdash \varphi \to \psi$.

Definition 3.2 *Let Θ be a first-order signature. A Θ-structure is a triple $\mathfrak{S} = \langle \mathbf{A}, S, \cdot^{\mathfrak{S}} \rangle$ such that $\mathbf{A}$ is a complete $H_3^{\vee,\triangle}$-algebra, and S is a non-empty set and $\cdot^{\mathfrak{S}}$ is an interpretation mapping defined on Θ as follows:*

1. *for each individual constant symbol c of Θ, $c^{\mathfrak{S}}$ of S,*
2. *for each function symbol f n-ary of Θ, $f^{\mathfrak{S}} : S^n \to S$,*
3. *for each predicate symbol P n-ary of Θ, $P^{\mathfrak{S}} : S^n \to A$.*

Given a Θ-structure $\mathfrak{S} = \langle \mathbf{A}, S, \cdot^{\mathfrak{S}} \rangle$, an $\mathfrak{S}$-valuation (or simply valuation) is a function $v : Var \to S$. Given $a \in S$ and $\mathfrak{S}$-valuation v, by $v[x \to a]$ we denote the following $\mathfrak{S}$-valuation, $v[x \to a](x) = a$ and $v[x \to a](y) = v(y)$ for any $y \in V$ such that $y \neq x$.

Let $\mathfrak{S} = \langle \mathbf{A}, S, \cdot^{\mathfrak{S}} \rangle$ be a Θ-structure and v an $\mathfrak{S}$-valuation. A Θ-structure $\mathfrak{S} = \langle \mathbf{A}, S, \cdot^{\mathfrak{S}} \rangle$ and an $\mathfrak{S}$-valuation v induce an interpretation map $||\cdot||^{\mathfrak{S}}_v$ for terms and formulas that can be defined as follows:

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$$||c||_v^{\mathfrak{S}} = c^{\mathfrak{S}}, \text{ if } c \in \mathcal{C}$$
$$||x||_v^{\mathfrak{S}} = v(x), \text{ if } x \in Var$$
$$||f(t_1, \cdots, t_n)||_v^{\mathfrak{S}} = f^{\mathfrak{S}}(||t_1||_v^{\mathfrak{S}}, \cdots, ||t_n||_v^{\mathfrak{S}}), \text{ for any } f \in \mathcal{F},$$
$$||P(t_1, \cdots, t_n)||_v^{\mathfrak{S}} = P^{\mathfrak{S}}(||t_1||_v^{\mathfrak{S}}, \cdots, ||t_n||_v^{\mathfrak{S}}), \text{ for any } P \in \mathcal{P},$$
$$||\alpha \to \beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \to ||\beta||_v^{\mathfrak{S}},$$
$$||\alpha \vee \beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \vee ||\beta||_v^{\mathfrak{S}},$$
$$||\triangle\alpha||_v^{\mathfrak{S}} = \triangle||\alpha||_v^{\mathfrak{S}},$$
$$||\forall x\alpha||_v^{\mathfrak{S}} = \bigwedge_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{S}},$$
$$||\exists x\alpha||_v^{\mathfrak{S}} = \bigvee_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{S}}.$$

We say that $\mathfrak{S}$ and v *satisfy* a formula φ, denoted by $\mathfrak{S} \vDash \varphi[v]$, if $||\varphi||_v^{\mathfrak{S}} = 1$. Besides, we say that φ is *true* in $\mathfrak{S}$ if $||\varphi||_v^{\mathfrak{S}} = 1$ for each $\mathfrak{S}$-valuation v and denoted by $\mathfrak{S} \vDash \varphi$. We say that φ is a *semantical consequence* of Γ in $\mathcal{QH}_3^{\vee,\triangle}$, if, for any structure $\mathfrak{S}$: if $\mathfrak{S} \vDash \gamma$ for each $\gamma \in \Gamma$, then $\mathfrak{S} \vDash \varphi$. For a given set of formulas Γ, we say that the structure $\mathfrak{S}$ is a *model* of Γ iff $\mathfrak{S} \vDash \gamma$ for each $\gamma \in \Gamma$.

Now, it is worth mentioning that the following property $||\varphi(x/t)||_v^{\mathfrak{S}} = ||\varphi||_{v[x\to||t||_v^{\mathfrak{S}}]}^{\mathfrak{S}}$ holds. Another important aspect of the definition of *semantical consequence* is that it is different to the propositional case because if we use the definition of valuation for this case, we are unable to prove an important rule as $\alpha(x) \vDash \forall x\alpha(x)$.

In addition, we need to recall an important property of complete $H_3^{\vee,\triangle}$-algebra.

Lemma 3.3 [16, Lemma 0.1.21] *Let A be a complete $H_3^{\vee,\triangle}$-algebra and the set $\{a_i\}_{i\in I}$ of element of A for any non-empty set I. Then if there exists $\bigvee_{i\in I} a_i$ ($\bigwedge_{i\in I} a_i$), then there exists $\bigvee_{i\in I} \triangle a_i$ ($\bigwedge_{i\in I} \triangle a_i$) and also, $\bigvee_{i\in I} \triangle a_i = \triangle \bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} \triangle a_i = \triangle \bigwedge_{i\in I} a_i$.*

This property is useful to prove the following theorem:

Theorem 3.4 *(Soundness Theorem) Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Theta$, if $\Gamma \vdash_\vee \varphi$ then $\Gamma \vDash \varphi$.*

Proof. In what follows we will consider an arbitrary but fixed structure $\mathfrak{S} = \langle A, S, \cdot^{\mathfrak{S}}\rangle$. It is clear that the propositional axioms are true in $\mathfrak{S}$. Now, we have to prove that the new axioms (Ax11) and (Ax12) are true in $\mathfrak{S}$, and the new inference rules (R3) and (R4) preserve trueness in $\mathfrak{S}$.

(Ax11) Suppose that φ is $\alpha(x/t) \to \exists x\alpha$. Then, $||\varphi||_v^{\mathfrak{S}} = ||\alpha||_{v[x\to||t||_v^{\mathfrak{S}}]}^{\mathfrak{S}} \to ||\exists x\alpha||_v^{\mathfrak{S}}$. It is clear that $||\alpha||_{v[x\to||t||_v^{\mathfrak{M}}]}^{\mathfrak{S}} \leq \bigvee_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{S}}$ and then, $||\alpha||_{v[x\to||t||_v^{\mathfrak{S}}]}^{\mathfrak{S}} \leq ||\exists x\alpha||_v^{\mathfrak{S}}$. Therefore $||\alpha(x/t) \to \exists x\alpha||_v^{\mathfrak{S}} = 1$ for every $\mathfrak{S}$-valuation v. (Ax12) is analogous to (Ax11). Now, according to Lemma 3.3, the axioms (Ax13) (Ax14) are true in $\mathfrak{S}$.

(R4) Let $\alpha \to \beta$ such that x is not free in α, and let $\alpha \to \forall x\beta$. Let us suppose that $||\alpha \to \beta||_v^{\mathfrak{S}} = 1$ for every $\mathfrak{S}$-valuation v. Now, consider a fix valuation v, then $||\alpha \to \forall x\beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \to ||\forall x\beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \to \bigwedge_{a\in S} ||\beta||_{v[x\to a]}^{\mathfrak{S}}$. On the other hand, by hypothesis, we know that $||\alpha||_u^{\mathfrak{S}} \leq ||\beta||_u^{\mathfrak{S}}$ for every $\mathfrak{S}$-valuation u. In particular, $||\alpha||_v^{\mathfrak{S}} = ||\alpha||_{v[x\to a]}^{\mathfrak{S}} \leq ||\beta||_{v[x\to a]}^{\mathfrak{S}}$ for every $\mathfrak{S}$-valuation v. Then, $||\alpha||_v^{\mathfrak{S}} \leq \bigwedge_{a\in S} ||\beta||_{v[x\to a]}^{\mathfrak{S}}$ and so, $||\alpha||_v^{\mathfrak{S}} \to \bigwedge_{a\in S} ||\beta||_{v[x\to a]}^{\mathfrak{S}} = 1$ for every $\mathfrak{S}$-valuation v. The proof of preservation of trueness for (R3) is analogous to (R4). $\square$

In what follows, we will prove a strong version of Completeness Theorem for $\mathcal{QH}_{\vee,\triangle}^3$ using the Lindenbaum-Tarski algebra in a similar way to the propositional case. Let us observe that the algebra of formulas is an absolutely free algebra generated by the atomic formulas and its quantified formulas.

Now, let us consider the relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash \alpha \to \beta$ and $\vdash \alpha \to \beta$, then we have the algebra $\mathfrak{Fm}_\Theta/\equiv$ is a $H_3^{\vee,\triangle}$-algebra and the proof is exactly the same as in the propositional case (see, for instance, [1]). On the other hand, it is clear that $\mathcal{QH}_3^{\vee,\triangle}$ is a Tarskian and finitary logic. So, we can consider the notion of (maximal) consistent and closed theories with respect to some formula in the same way as the propositional case. Therefore, we have that Lindenbaum- Łos' Theorem holds for $\mathcal{QH}_3^{\vee,\triangle}$. Then, we have the following:

Theorem 3.5 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Theta$, with Γ non-trivial maximal respect to φ in $\mathcal{QH}_3^{\vee,\triangle}$. Let $\Gamma/\equiv = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of $\mathfrak{Fm}_\Theta/\equiv$, then:*

1. *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$, then $\beta \in \Gamma$. Besides, it is verified that $\Gamma/\equiv = \{\overline{\alpha} : \Gamma \vdash \alpha\}$, which, in this case, we say it is closed.*

2. *$\Gamma/\equiv$ is a modal deductive system of $\mathfrak{Fm}_\Theta/\equiv$. Also, if $\overline{\varphi} \notin \Gamma/\equiv$ and for any modal deductive system $\overline{D}$ which is closed in the sense of 1 and properly contains to $\Gamma/\equiv$, then $\overline{\varphi} \in \overline{D}$.*

Proof. According to the proof of Theorem 2.20, we only have to consider the rules $(R3)$ and $(R4)$. The fact that $\Gamma/\equiv$ is closed follows immediately.

In order to complete the proof, we have to consider two new cases 5 and 6. It is clear that $\Gamma/\equiv$ is a subset of $\overline{D}$. Now, let us consider $\overline{\phi} \in \overline{D}$ then $\overline{\phi} \notin \Gamma/\equiv$ and remember $D = \{\alpha : \overline{\alpha} \in \overline{D}\}$.

Case 5: There exists $\{j, t_1, ..., t_m\} \subseteq \{1, ..., k-1\}$ such that $\alpha_{t_1}, ..., \alpha_{t_m}$ is a derivation of $\alpha_j = \theta \to \beta$. Let us suppose that $\alpha_n = \exists x\theta \to \beta$ is obtained by α_j applying $(R3)$. From induction hypothesis, we have that $\overline{\theta \to \beta} \in \overline{D}$. From the latter, we obtain $\overline{\exists x\theta \to \beta} \in \overline{D}$.

Case 6: There exists $\{j, t_1, ..., t_m\} \subseteq \{1, ..., k-1\}$ such that $\alpha_{t_1}, ..., \alpha_{t_m}$ is a derivation of $\alpha_j = \theta \to \beta$. Let us suppose that $\alpha_n = \theta \to \forall x\beta$ is obtained by α_j applying $(R4)$. From induction hypothesis, we have $\overline{\theta \to \beta} \in \overline{D}$ and then, $\overline{\theta \to \forall x\beta} \in \overline{D}$. □

We note that for a given maximal consistent theory Γ of $\mathfrak{Fm}_\Theta$ we have $\Gamma/\equiv$ is a maximal modal deductive system of $\mathfrak{Fm}_\Theta/\equiv$. By well-known results of Universal Algebras, if we denote $A := \mathfrak{Fm}_\Theta/\equiv$ and $\theta := \Gamma/\equiv$, we have the quotient algebra A/θ is a simple algebra, see Corollary 2.13. From the latter and by adapting the first isomorphism theorem for Universal Algebras, we have that A/θ is isomorphic to $\mathfrak{Fm}_\Theta/\Gamma$ which is defined by the congruence $\alpha \equiv_\Gamma \beta$ iff $\alpha \to \beta, \beta \to \alpha \in \Gamma$.

Theorem 3.6 *(Completeness Theorem) Let $\Gamma \cup \{\varphi\}$ be a set of sentences, then $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. Let us suppose $\Gamma \vDash \varphi$ and $\Gamma \nvDash \gamma$. Then, by Lindenbaum- Łos' Lemma, there exists Δ maximal consistent theory such that $\Gamma \subseteq \Delta$. Now, consider the algebra $\mathfrak{Fm}_\Theta/\Delta$ defined by the congruence $\alpha \equiv_\Delta \beta$ iff $\alpha \to \beta, \beta \to \alpha \in \Delta$. In view of the above observations, we know that $\mathfrak{Fm}_\Theta/\Delta$ is isomorphic to a subalgebra of $\mathbb{C}_3^{\to,\vee}$ and so, complete as lattice.

Now, let us take the canonical projection $\pi_\Delta : \mathfrak{Fm} \to \mathfrak{Fm}_\Theta/\Delta$ defined by $\pi_\Delta(\alpha) = |\alpha|$ where $|\alpha|$ denotes the equivalence class of $\alpha \in \mathfrak{Fm}$. In this sense, consider the structure $\mathfrak{M} = \langle \mathfrak{Fm}_\Theta/\Delta, T_\Theta, \cdot^{T_\Theta} \rangle$ where T_Θ is a set of terms. It is clear that for every $t \in T_\Theta$ we have an associated constant $\hat{t}$ of Θ. Now, let us take a function $\mu : Var \to T_\Theta$ defined by $\mu(x) = x$. Then, we have the interpretation $||\cdot||_\mu^{\mathfrak{M}} : \mathfrak{Fm} \to \mathfrak{Fm}_\Theta/\Delta$ defined by if $\hat{t}$ is a constant, then $||\hat{t}||_\mu^{\mathfrak{M}} := t$; if $f \in \mathcal{F}$, then $||f(t_1, \cdots, t_n)||_\mu^{\mathfrak{M}} = f(t_1, \cdots, t_n)$; if $P \in \mathcal{P}$, then $||P(t_1, \cdots, t_n)||_\mu^{\mathfrak{M}} = \pi_\Delta(P(t_1, \cdots, t_n))$. Our interpretation is defined for atomic formulas but it is easy to see that $||\alpha||_\mu^{\mathfrak{M}} = \pi_\Delta(\alpha)$ for every quantifier-free formula α. Moreover, it is easy to see that for every formula $\phi(x)$ and every term t, we have $||\phi(x/\hat{t})||_\mu^{\mathfrak{M}} = ||\phi(x/t)||_\mu^{\mathfrak{M}}$. Therefore, from the latter property and by (Ax12) and (R4), we have $||\forall x\alpha||_\mu^{\mathfrak{M}} = \bigwedge_{a \in T_\Theta} ||\alpha||_{\mu[x\to a]}^{\mathfrak{M}}$ and now using (Ax11) and (R3), we obtain $||\exists x\alpha||_\mu^{\mathfrak{M}} = \bigvee_{a \in T_\Theta} ||\alpha||_{\mu[x\to a]}^{\mathfrak{M}}$. So, $||\cdot||_\mu^{\mathfrak{M}}$ is an interpretation map such that $||\alpha||_\mu^{\mathfrak{M}} = 1$ iff $\alpha \in \Delta$. On the other hand, it is not hard to see for every closed formula (sentence) β, we have $||\beta||_\mu^{\mathfrak{M}} = ||\beta||_v^{\mathfrak{M}}$ for every $\mathfrak{M}$-valuation v. Therefore, $\mathfrak{M} \vDash \gamma$ for every $\gamma \in \Gamma$ but $\mathfrak{M} \nvDash \varphi$ which is a contradiction. □

Given a formula φ and suppose $\{x_1, \cdots, x_n\}$ is the set of variables of φ, the *universal closure* of φ is defined by $\forall x_1 \cdots \forall x_n \varphi$. Thus, it is clear that if φ is a sentence, then the universal closure of φ is itself. Now, we are in condition to prove the following Completeness Theorem for formulas:

Theorem 3.7 *Let $\Gamma \cup \{\varphi\}$ be a set of formulas, then $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. Let us suppose $\Gamma \vDash \varphi$ and consider the set $\forall\Gamma$ all universal closure of Γ. From the latter and definition of $\vDash$, we have $\forall\Gamma \vDash \forall x_1 \cdots \forall x_n \varphi$. Then, according to Theorem 3.6, $\forall\Gamma \vdash \forall x_1 \cdots \forall x_n \varphi$. Now, from the latter, (Ax12) and (R4), we have $\Gamma \vdash \varphi$ as desired. $\square$

4 Final Comments and Future Work

As final comments, we can say that our proof of the Completeness Theorem is different from the ones we can find in the literature (see for instance [1, 25]) because we use an algebraic technique developed by A. Monteiro, [13]. This technique can be used in the class studied in [6]. Indeed, consider the class of 3-valued modal Hilbert algebra ($\triangle H_3$-algebras) of Definition 2.2. From Lemma 2.11, it is possible to see that this class constitutes a semisimple variety. Now, let us consider the logic $\triangle\mathcal{H}_3$ over the signature $\{\to, \triangle\}$ defined by the axiom schemas (Ax1) to (Ax3) and (Ax7) to (Ax10), as well as the rules (MP) and (NEC). Taking in mind, the corresponding definitions of Section 2.1, it is possible to prove the following theorem:

Theorem 4.1 *(Soundness and Completeness of $\triangle\mathcal{H}_3$ w.r.t. the class of $\triangle H_3$-algebras) Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, $\Gamma \vdash_{\triangle\mathcal{H}_3} \varphi$ if and only if $\Gamma \vDash_{\triangle\mathcal{H}_3} \varphi$.*

Now, consider the first order version of $\triangle\mathcal{H}_3$ that we denote $\mathcal{Q}\triangle\mathcal{H}_3$. For $\mathcal{Q}\triangle\mathcal{H}_3$ we use the axioms (Ax11), (Ax12), rules of Definition 3.1 and notation of Section 3. Then, we have the following Theorem:

Theorem 4.2 *Let $\Gamma \cup \{\varphi\}$ be a set of formulas of $\mathcal{Q}\triangle\mathcal{H}_3$, then $\Gamma \vDash_{\mathcal{Q}\triangle\mathcal{H}_3} \varphi$ if only if $\Gamma \vdash_{\triangle H_3} \varphi$.*

The two last Theorems can be proved in the same way as the corresponding ones of the logic $\mathcal{H}^3_{\vee,\triangle}$ and $\mathcal{Q}\mathcal{H}_3^{\vee,\triangle}$. Yet this technique can not be applied to any logics from non-semisimple varieties, such as (n-valued) Heyting algebras, MV-algebras, Hilbert algebras, residuated lattices and so on.

As future work, we will present a study of logics from semisimple varieties of algebras studied in the Monteiro's school. All these systems will allow us to apply the technique presented in this paper.

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