On the Paraconsistent Logic $\mathbf{CG}'_3$

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Abstract. Paraconsistent logical systems are well-known reasoning frameworks aimed to infer new facts or properties under contradictory assumptions. Applications of these systems are well known in a wide range of computer science domains. In this article, we study the paraconsistent logic $\mathbf{CG}'_3$, which can be viewed as an extension of the logic $\mathbf{G}'_3$. $\mathbf{CG}'_3$ is also 3-valued, but with two designated values. Main results can be summarized as follows: a Hilbert-type axiomatization, based on Kalmár's approach; and a new notion of validity, based on also novel Kripke semantics.

Keywords. Many-valued logic, paraconsistent logic, Kripke-type semantics, Hilbert calculi, $\mathbf{CG}'_3$.

1 Introduction

Many valued logics, as classical ones, are based on the principle of truth-functionality.[1] In the classical approach, there are only two truth values, *falsum* and *verum*, which are commonly written "0" and "1". Contrastingly, in many-valued logics more than two truth values are considered. A survey on many-valued logics may be found in [9]. Originally motivated by philosophical aims, many-valued logics are also inspired by formal technical concerns regarding functional completeness.

Among the first applications of many-valued logics, one may found hardware design. Analogously, as classical logic is used as a technical tool for the analysis and synthesis of electrical circuits built up from switches with two stable states,

denoting voltage levels, many-valued logic can be used as general model of electrical circuits with more than two stable states. This application field of many-valued logic is called many-valued switching [6].

We now list some current state-of-the-art applications of many-valued logics in artificial intelligence: imprecise notions inherently tied to commonsense reasoning in expert systems, can be naturally modeled via fuzzy logic. Inference systems for many-valued logics, fuzzy logic in this context, can then be used as reasoning frameworks in expert systems [9]. A relatively recent research perspective in the AI setting, concerns the many-valued generalization of description logics, well-known as the reasoning foundation of the semantic web [10].

In [14], it is extensively reported on non–monotonic reasoning based on paraconsistent logics. In particular, it is proposed a logic programming semantics based on the paraconsistent logic $\mathbf{G}'_3$. This is called $\mathbf{G}'_3$–stable semantics. Inconsistent and vague domains can be naturally modeled with this $\mathbf{G}'_3$–stable semantics.

In [16], Priest affirms that one of the motives of da Costa, to build the paraconsistent logic $\mathbf{C}_\omega$, was dualizing the negation of intuitionistic logic. Intuitionistic logic is a logic that allows for "truth-value gaps"; for example, the Law Excluded Middle fails. The logic $\mathbf{C}_\omega$ achieves this but with explicit costs; for example, the substitution of provable equivalents fails. Da Costa
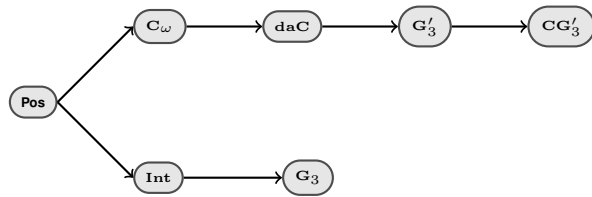
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[1] The truth of a compound sentence is obtained by the truth values of its component sentences.

Fig. 1. Comparison of $\mathbf{G}_3'$ and $\mathbf{CG}_3'$ with other logics

proceeded axiomatically, preserving the positive part of intuitionistic logic, and changing the axioms of negation.

But the various semantics for intuitionistic logic suggest other ways of pursuing da Costa's goal. Evidence of this is the paraconsistent logic created by Priest that arises when dualizing the modeling conditions for the negation in Kripke semantics for intuitionistic logic. This new system is called da Costa logic $\mathbf{daC}$. We can find at the end of [7, section 2] a brief study of extensions of fragments of Heyting Brouwer Logic. This is the case of the family of logics $\mathbf{daCG_n}$; each an extension of $\mathbf{daC}$ characterized by a Kripke frame for $\mathbf{daC}$, which is linearly ordered and has $n-1$ points. We have that $\mathbf{G}_3'$ corresponds to $\mathbf{daCG_3}$, and clearly, the characterization agrees.

In [15], Osorio et al. define $\mathbf{G}_3'$ through its multi-valued semantics. $\mathbf{CG}_3'$ is an extension of $\mathbf{G}_3'$ [13]. In contrast with $\mathbf{G}_3'$, whose designated value is 1, $\mathbf{CG}_3'$ has 1 and 2 as designated values. It is important to note that $\mathbf{G}_3'$ is not comparable with Gödel logic $\mathbf{G}_3$. In Figure 1, we present some logics related to $\mathbf{CG}_3'$ where the arrows mean contention.

The structure of the document is as follows. In Section 2, we present the definition of $\mathbf{CG}_3'$ from many-valued semantics. In Section 3, we give the formal axiomatic theory $\mathbb{L}$ for $\mathbf{CG}_3'$ and examine some interesting properties of $\mathbb{L}$, and we close the section seeing that $\mathbb{L}$ is sound and complete concerning $\mathbf{CG}_3'$. To prove that $\mathbb{L}$ is complete, the authors follow the procedure of completeness proof used for classical logic given in [12] and originally due to Kalmar. This method has been used in other many-valued logics, see [1, 11]. In Section 4, we show the semantical similarities between $\mathbf{CG}_3'$ and the many-valued logic Ł3. In Section 5, we

introduce the Kripke-type semantics to $\mathbf{CG}_3'$, in two different ways.

2 Many-Valued Semantics for $\mathbf{CG}_3'$

We first introduce the syntax of the logical formulas considered in this paper. We follow standard notation and basic definitions as M. Osorio in [15].

The following symbols will be used for logical connectives: $\wedge$ (conjunction, binary); $\vee$ (disjunction, binary); $\leftrightarrow$ (biconditional, binary); $\neg$ (weak negation, unary); ∇ (inconsistency operator, unary); $\sim$ (strong negation, unary) and $\perp$ (bottom formula, 0-arity).

Fix the propositional language $\mathcal{L}$ whose primitive symbols are:

— the variables $p_0, p_1, \ldots$;

— the connectives: $\wedge, \vee, \neg$, and $\rightarrow$;

— the punctuation marks: (and),

— the formulas of $\mathcal{L}$ are defined inductively:

  – all the variables in $\mathcal{L}$ are atomic formulas o simply atoms;

  – if φ and ψ are formulas then $(\varphi \wedge \psi), (\varphi \vee \psi)$, and $(\varphi \rightarrow \psi)$ are also formulas.

One of the most popular semantics for many-valued logical systems is the standard logical matrices. The most appropriate way to define semantics for a logic of many-values is through a logical matrix characteristic from its language, that is:

— the set of values of truth (domain),

— the set of designated values, which form a subset of the set of truth degrees and act as substitutes for the traditional truth value verum,

— the functions of degree of truth interpreted by the propositional connectives.

Table 1. Truth functions of the connectives in $\mathbf{CG}'_3$

$\vee$	**0**	**1**	**2**
0	0	1	2
1	1	1	2
2	2	2	2

$\wedge$	**0**	**1**	**2**
0	0	0	0
1	0	1	1
2	0	1	2

$\to$	**0**	**1**	**2**
0	2	2	2
1	0	2	2
2	0	1	2

$\neg$	
0	2
1	2
2	0

(a) Connectives $\vee, \wedge, \to$, and $\neg$

	$\neg\neg$	∇	$\sim$
0	0	0	2
1	0	2	0
2	2	0	0

$\leftrightarrow$	**0**	**1**	**2**
0	2	0	0
1	0	2	1
2	0	1	2

(b) Connectives $\neg\neg, \nabla, \sim$, and $\leftrightarrow$

A well-formed formula φ of a propositional language counts as valid under some valuation v[2] if and only if it has a designated truth value under v. And φ is a tautology if and only if it is valid under all valuations, and we denote this by $\models \varphi$.

The paraconsistent logic $\mathbf{CG}'_3$ is introduced in [13] and is given, by the matrix $M = \langle D, D^*, F \rangle$; where $D = \{0, 1, 2\}$ is the domain, $D^* = \{1, 2\}$ is the set of designated values, and F is the set of truth functions for the connectives $\{\wedge, \vee, \to, \neg\}$ and consists of the functions displayed in Table 1a.

Definition 1. *Given a formula φ in the language of a logic $\mathbf{CG}'_3$, we say that this is a tautology in $\mathbf{CG}'_3$ if, for every possible valuation, the formula φ is valid, and we denote this by $\models_{\mathbf{CG}'_3} \varphi$.*

In [8], the authors present some properties from the semantic point of view that veri-

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fies this logic, to mention some we have:

WE $\neg\varphi \to (\neg\neg\varphi \to \psi)$
NI $\neg\neg(\varphi \to \varphi)$
WC $(\neg\varphi \to \neg\psi) \leftrightarrow (\neg\neg\psi \to \neg\neg\varphi)$
Peirce $\left((\varphi \to \psi) \to \varphi\right) \to \varphi$

3 Axiomatization of $\mathbf{CG}'_3$ Logic

Let us consider $\mathbb{L}$, a formal axiomatic theory for $\mathbf{CG}'_3$ defined over the signature $\mathcal{L} = \{\neg, \to, \wedge\}$. Some logical connectives defined in terms of the primitives:

$$
\begin{aligned}
\sim\varphi &:= \varphi \to (\neg\varphi \wedge \neg\neg\varphi) \\
\nabla\varphi &:= \sim\sim\varphi \wedge \neg\varphi \\
\varphi \vee \psi &:= \left((\varphi \to \psi) \to \psi\right) \wedge \left((\psi \to \varphi) \to \varphi\right) \\
\varphi \leftrightarrow \psi &:= (\varphi \to \psi) \wedge (\psi \to \varphi)
\end{aligned}
$$

The truth tables of the connective $\sim$, ∇, and $\leftrightarrow$ can see in Table 1b. The set of atoms is denoted as $Atom(\mathcal{L})$, and the set of well-formed formulas constructed in the usual way and denoted by $Form(\mathcal{L})$.

Axiom Schemes

Pos1: $\varphi \to (\psi \to \varphi)$

Pos2: $\left(\varphi \to (\psi \to \sigma)\right) \to \left((\varphi \to \psi) \to (\varphi \to \sigma)\right)$

Pos3: $(\varphi \wedge \psi) \to \varphi$

Pos4: $(\varphi \wedge \psi) \to \psi$

Pos5: $\varphi \to \left(\psi \to (\varphi \wedge \psi)\right)$

Cw1: $\varphi \vee \neg\varphi$

CG-1: $\left((\varphi \to \psi) \to \varphi\right) \to \varphi$

CG-2: $\neg\neg(\varphi \to \psi) \leftrightarrow \left((\varphi \to \psi) \wedge (\neg\neg\varphi \to \neg\neg\psi)\right)$

CG-3: $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$

CG-4: $\neg\varphi \to (\neg\neg\varphi \to \psi)$

Inference Rule

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \mathbf{MP}$$

We say that φ is derivable from Γ in $\mathbb{L}$, denoted as $\Gamma \vdash_{\mathbb{L}} \varphi$ if there is a derivation of φ of Γ in $\mathbb{L}$.

As can be seen, the list of axioms given above contains only the first five axioms of the positive part of Intuitionistic Logic, in addition to $\mathbf{Cw}1$, $\mathbf{CG\text{-}1}$, $\mathbf{CG\text{-}2}$, $\mathbf{CG\text{-}3}$, and the axiom $\mathbf{CG\text{-}4}$.

The following meta-theorems of $\mathbb{L}$ will prove to be quite useful, their proofs are straightforward.

Theorem 1. *Let Γ, Δ be set of formulas, and let φ, ψ be formulas, then the following properties it holds in $\mathbb{L}$:*

1. **Monotonicity**:
   If $\Gamma \vdash_{\mathbb{L}} \varphi$ then $\Gamma \cup \Delta \vdash_{\mathbb{L}} \varphi$.

2. **Deduction theorem**:
   $\Gamma, \varphi \vdash_{\mathbb{L}} \psi$ if and only if $\Gamma \vdash_{\mathbb{L}} \varphi \to \psi$.

3. **Cut**:
   If $\Gamma \vdash_{\mathbb{L}} \varphi$ and $\Delta, \varphi \vdash_{\mathbb{L}} \psi$ then $\Gamma \cup \Delta \vdash \psi$.

4. **Rules-AND**:
   $\Gamma \vdash_{\mathbb{L}} \varphi \wedge \psi$ if and only if $\Gamma \vdash_{\mathbb{L}} \varphi$ and $\Gamma \vdash_{\mathbb{L}} \psi$.

Lemma 1. *For any formulas φ, ψ, σ, and ξ, the following formulas are theorems in $\mathbb{L}$:*

(a) $\vdash \varphi \to \varphi$

(b) $\varphi \to \psi, \psi \to \sigma \vdash \varphi \to \sigma$

(c) $\varphi \to \psi, \sigma \to \xi \vdash (\varphi \wedge \sigma) \to (\psi \wedge \xi)$

(d) $\vdash \left(\varphi \to (\psi \to \gamma) \right) \to \left((\varphi \wedge \psi) \to \gamma \right)$

(e) $\varphi \to (\psi \to \gamma) \vdash \psi \to (\varphi \to \gamma)$

(f) $\vdash (\varphi \wedge \psi) \leftrightarrow (\psi \wedge \varphi)$

(g) $\vdash \varphi \to \left((\varphi \to \psi) \to \psi \right)$

(h) $\varphi \to \sigma, (\varphi \to \psi) \to \sigma, \sigma \to \psi \vdash \sigma$

Proof. Each item can be proved using $\mathbf{Pos}1\text{-}\mathbf{Pos}5$, **MP**, and Deduction theorem. $\square$

It, is worth mentioning that in the list of axiom schemes of $\mathbb{L}$ not all axioms of the positive part of Intuitionistic logic are included, however, they can be derived from this list, as well as from other well-known axiom schemes some of them are shown in the following lemma:

Lemma 2. *The following formulas are theorems in $\mathbb{L}$:*

Pos6	$\varphi \to (\varphi \vee \psi)$
Pos7	$\psi \to (\varphi \vee \psi)$
Pos8	$(\varphi \to \sigma) \to \left((\psi \to \sigma) \to ((\varphi \vee \psi) \to \sigma) \right)$
Cw2	$\neg\neg\varphi \to \varphi$
E1	$(\neg\varphi \to \neg\psi) \leftrightarrow (\neg\neg\psi \to \neg\neg\varphi)$
ON	$\neg\varphi \leftrightarrow \neg\neg\neg\varphi$
CG$'$3	$\nabla\varphi \to \varphi$

Proof. We only present the proof of $\mathbf{CG}'3$, the other formulas, are proved using the axiom schemes, Lemma 1, and Modus Ponens.

1.	$\nabla\varphi$	Hyp
2.	$\sim\sim\varphi \wedge \neg\varphi$	Abb. ∇
3.	$(\sim\sim\varphi \wedge \neg\varphi) \to \sim\sim\varphi$	**Pos**3
4.	$\sim\sim\varphi$	1, 2, **MP**
5.	$\sim\varphi \to (\neg\sim\varphi \wedge \neg\neg\sim\varphi)$	Abb. $\sim$
6.	$(\varphi \to (\neg\varphi \wedge \neg\neg\varphi)) \to$ $(\neg\sim\varphi \wedge \neg\neg\sim\varphi)$	Abb. $\sim$
7.	$\neg\sim\varphi \to (\neg\neg\sim\varphi \to \varphi)$	**CG-4**
8.	$(\neg\neg\varphi \wedge \neg\neg\sim\varphi) \to \varphi$	Lemma 1
9.	$(\varphi \to (\neg\varphi \wedge \neg\neg\varphi)) \to \varphi$	Lemma 1
10.	$((\varphi \to (\neg\varphi \wedge \neg\neg\varphi)) \to \varphi) \to \varphi$	**CG-1**
11.	φ	9, 10, **MP**
12.	$\nabla\varphi \vdash \varphi$	1-11
13.	$\nabla\varphi \to \varphi$	**DMT**
		$\square$

Theorem 2. *Let Γ be a set of formulas and be φ, ψ arbitrary formulas, then the following property* **Proof-by-cases**, *it is fulfilled in $\mathbb{L}$.*

$\Gamma, \varphi \vdash_{\mathbb{L}} \psi$ *and* $\Gamma, \neg\varphi \vdash_{\mathbb{L}} \psi$ *if and only if* $\Gamma \vdash_{\mathbb{L}} \psi$.

Proof. Suppose that $\Gamma, \varphi \vdash_{\mathbb{L}} \psi$ and $\Gamma, \neg\varphi \vdash_{\mathbb{L}} \psi$. Using the Deduction theorem, we have that, $\Gamma \vdash_{\mathbb{L}} \varphi \to \psi$ and $\Gamma \vdash_{\mathbb{L}} \neg\varphi \to \psi$, applying $\mathbf{Pos}8$ we obtain $\Gamma \vdash_{\mathbb{L}} (\varphi \vee \neg\varphi) \to \psi$. Finally, using the axiom $\mathbf{Cw}1$ and **MP**, we have $\Gamma \vdash_{\mathbb{L}} \psi$, as required. $\square$

3.1 Soundness and Completeness Theorem

Now it is proved that $\mathbf{CG}'_3$ is sound concerning $\mathbb{L}$, that is, the theorems in $\mathbb{L}$ are tautologies in $\mathbf{CG}'_3$.

Theorem 3 (**Soundness of** $\mathbb{L}$). *Let φ be a formula. If φ is a theorem in $\mathbb{L}$, then φ is a tautology in $\mathbf{CG}'_3$, that is if $\vdash_\mathbb{L} \varphi$ then $\models_{\mathbf{CG}'_3} \varphi$.*

Proof. Each axiom scheme of $\mathbb{L}$ evaluates to 1 or 2, according to the tables of $\mathbf{CG}'_3$, that is each axiom scheme is a tautology in $\mathbf{CG}'_3$. It remains to see that **MP** preserves tautologies. Suppose that ψ and $\psi \to \gamma$ are tautologies, and γ takes the value 0 for some 3-valuation. Since ψ is a tautology, it must take 1 or 2. Therefore, $\psi \to \gamma$ is forced to take the value 0 for that valuation. This last contradicts the assumption that $\psi \to \gamma$ is a tautology. Therefore γ never takes the value 0. □

To prove the lemma 5, which is imperative to prove the completeness theorem, the Lemma 4 is needed, whose proof needs any supplementary results, viz, Proposition 1 and Lemma 3. These lemmas model the behavior of the connective $\sim$, $\neg$ and ∇. The proof of each item is straightforward and employs the axiom schemes and the Modus Ponens rule.

Proposition 1. *For any formulas φ, ψ, the following formulas are theorems in $\mathbb{L}$:*

(a) $\vdash \sim\varphi \to (\varphi \to \psi)$

(b) $(\varphi \to \psi), (\varphi \to \sim\psi), \varphi \vdash (\neg\psi \wedge \neg\neg\psi)$

(c) $\vdash (\neg\psi \wedge \neg\neg\psi) \to (\neg\varphi \wedge \neg\neg\varphi)$

(d) $\vdash (\varphi \to \psi) \to \big((\varphi \to \sim\psi) \to \sim\varphi\big)$

(e) $\vdash (\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$

(f) $\sim\sim\psi, \sim\varphi, (\psi \to \varphi) \vdash (\sim\psi \wedge \sim\sim\psi)$

(g) $\vdash (\sim\psi \wedge \sim\sim\psi) \to \big(\neg(\psi \to \varphi) \wedge \neg\neg(\psi \to \varphi)\big)$

(h) $\vdash (\sim\sim\psi \wedge \sim\varphi) \to \sim(\psi \to \varphi)$

(i) $\vdash \sim\sim\varphi \to \sim\sim(\psi \to \varphi)$

(j) $\vdash \varphi \to \sim\sim\varphi$

(k) $\vdash \psi \to \big(\sim\varphi \to \sim(\psi \to \varphi)\big)$

(l) $\vdash \sim\sim\varphi \to \varphi$

(m) $\vdash (\sim\sim\varphi \wedge \sim\sim\psi) \to \sim\sim(\varphi \wedge \psi)$

The following lemma characterizes the behavior of negation $\neg$ and negation $\sim$.

Lemma 3. *The following formulas are theorems in $\mathbb{L}$:*

(a) $\vdash \sim\varphi \to \neg\varphi$

(b) $\vdash \sim\neg\varphi \to \sim\sim\varphi$

(c) $\vdash \neg\neg\varphi \to \sim\sim\varphi$

(d) $\vdash \sim\varphi \to \neg\neg\sim\varphi$

(e) $\vdash \neg\sim\varphi \to \sim\sim\varphi$

Now we present Lemma 4, which models the behavior of the connectives of $\mathbb{L}$. If a formula $v(\varphi) = 0$, then $\sim\varphi$ is assigned. When $v(\varphi) = 1$, $\nabla(\varphi)$ is assigned, while $\neg\neg\varphi$ corresponds to the case where $v(\varphi) = 2$. With these ideas, the interpretation of item (a) is as follows: $\neg\neg\varphi$ tells us that the value of the formula φ is 2, then the meaning of its negation $\neg\varphi$ must be 0, which is written by $\sim\neg\varphi$. Item (b) indicates that if φ evaluates 1, then we have $\nabla\varphi$, and its negation must be 2. Finally, item (c) models the fact that when a formula takes the value 2, its negation must be equal to 0. These items model the connective negation. The connective implication is modeling by items (d) to (i), and the connective conjunction model by entry (j) to (o).

Lemma 4. *The following formulas are theorems in $\mathbb{L}$:*

(a) $\vdash \neg\neg\varphi \to \sim\neg\varphi$

(b) $\vdash \nabla\varphi \to \neg\neg\neg\varphi$

(c) $\vdash \sim\varphi \to \neg\neg\neg\varphi$

(d) $\vdash \sim\varphi \to \neg\neg(\varphi \to \psi)$

(e) $\vdash \neg\neg\psi \to \neg\neg(\varphi \to \psi)$

(f) $\vdash (\nabla\varphi \wedge \sim\psi) \to \sim(\varphi \to \psi)$

(g) $\vdash (\nabla\varphi \wedge \nabla\psi) \to \neg\neg(\varphi \to \psi)$

(h) $\vdash (\neg\neg\varphi \wedge \nabla\psi) \to \nabla(\varphi \to \psi)$

(i) $\vdash (\neg\neg\varphi \wedge \sim\psi) \to \sim(\varphi \to \psi)$

(j) $\vdash \sim\varphi \to \sim(\varphi \wedge \psi)$

(k) $\vdash \sim\psi \to \sim(\varphi \wedge \psi)$

(l) $\vdash (\nabla\varphi \wedge \nabla\psi) \to \nabla(\varphi \wedge \psi)$

(m) $\vdash (\nabla\varphi \wedge \neg\neg\psi) \to \nabla(\varphi \wedge \psi)$

(n) $\vdash (\neg\neg\varphi \wedge \nabla\psi) \to \nabla(\varphi \wedge \psi)$

(o) $\vdash (\neg\neg\varphi \wedge \neg\neg\psi) \to \neg\neg(\varphi \wedge \psi)$

Now it is shown that $\mathbf{CG}'_3$ is complete concerning $\mathbb{L}$. To prove that each tautology in $\mathbf{CG}'_3$ is a theorem in $\mathbb{L}$, the completeness proof strategy used for the Classic Propositional Logic given in [12] originally due to Kalmár.

Definition 2. *Given a 3-valuation v of $\mathbf{CG}'_3$ and a formula φ, we define the formula φ_v called the image of φ, as follows:*

$$\varphi_v = \begin{cases} \neg\neg\varphi & \text{if } v(\varphi) = 2, \\ \nabla\varphi & \text{if } v(\varphi) = 1, \\ \sim\varphi & \text{if } v(\varphi) = 0. \end{cases}$$

Let Φ be a set of formulas. The set $\{\varphi_v | \varphi \in \Phi\}$ is denoted by Φ_v.

Lemma 5 (**Kalmár's Lemma for $\mathbf{CG}'_3$**). *Let be φ a formula and v a valuation in $\mathbf{CG}'_3$, if $Atom(\varphi)$ denotes the set of formulas in φ, then $Atom(\varphi)_v \vdash \varphi_v$.*

Proof. The proof is done by induction on the complexity of φ.

Base Case: If $\varphi = p$, where p is an atomic formula, then we need to show that $Atom(\varphi)_v \vdash \varphi_v$, but this is evident since $Atom(\varphi)_v = \varphi_v = p_v$.

Let us see now that for any formula φ, the claim is true. Suppose that if formula ψ has less complexity than φ, then the lemma holds.

Inductive step: Suppose that φ is a non-atomic formula. We have three cases, and we only present the implication case:

Case $\to$: Suppose that $\varphi = \beta \to \zeta$. By the inductive hypothesis, we know that $Atom(\beta)_v \vdash \beta_v$ and $Atom(\zeta)_v \vdash \zeta_v$. Then, we have six subcases:

1. If $v(\beta) = 0$, then $\beta_v = \sim\beta$. By inductive hypothesis, $Atom(\beta)_v \vdash \sim\beta$. Note that $v(\varphi) = v(\beta \to \zeta) = 2$, so $\varphi_v = \neg\neg\varphi$. But $\varphi = \beta \to \zeta$, hence $\varphi_v = \neg\neg(\beta \to \zeta)$. We need to prove $Atom(\varphi)_v \vdash \neg\neg(\beta \to \zeta)$. By Lemma 4, we know that $\vdash \sim\beta \to \neg\neg(\beta \to \zeta)$ and by inductive hypothesis, $Atom(\beta)_v \vdash \sim\beta$. With the application of **MP** to previous statements, we conclude $Atom(\beta)_v \vdash \neg\neg(\beta \to \zeta)$. Finally, by monotonicity $Atom(\varphi)_v \vdash \neg\neg(\beta \to \zeta)$.

2. If $v(\zeta) = 2$, then $\zeta_v = \neg\neg\zeta$. By hypothesis: $Atom(\zeta)_v \vdash \neg\neg\zeta$. Note that $v(\varphi) = v(\beta \to \zeta) = 2$, so $\varphi_v = \neg\neg\varphi$. But $\varphi = \beta \to \zeta$, then $\varphi_v = \neg\neg(\beta \to \zeta)$. We need to prove $Atom(\varphi)_v \vdash \neg\neg(\beta \to \zeta)$. By Lemma 4, we know that $\vdash \neg\neg\zeta \to \neg\neg(\beta \to \zeta)$ and by inductive hypothesis, $Atom(\zeta)_v \vdash \neg\neg\zeta$. Applying **MP** to previous steps, we conclude $Atom(\zeta)_v \vdash \neg\neg(\beta \to \zeta)$. Finally, by monotonicity $Atom(\varphi)_v \vdash \neg\neg(\beta \to \zeta)$.

3. If $v(\beta) = 1$ and $v(\zeta) = 0$, then $\beta_v = \nabla\beta$ and $\zeta_v = \sim\zeta$. By inductive hypothesis, we have that $Atom(\beta)_v \vdash \nabla\beta$ and $Atom(\zeta)_v \vdash \sim\zeta$. Note that $v(\varphi) = v(\beta \to \zeta) = 0$. So $\varphi_v = \sim\varphi$, then $\varphi_v = \sim(\beta \to \zeta)$. We need to prove $Atom(\varphi)_v \vdash \sim(\beta \to \zeta)$. By Lemma 4, we know that $\vdash (\nabla\beta \wedge \sim\zeta) \to \sim(\beta \to \zeta)$ and by inductive hypothesis, monotonicity and Rules-AND we have: $Atom(\varphi)_v \vdash \nabla\beta \wedge \sim\zeta$. With the application of **MP** to previous steps, we conclude $Atom(\varphi)_v \vdash \sim(\beta \to \zeta)$.

4. If $v(\beta) = 1$ and $v(\zeta) = 1$,then $\beta_v = \nabla\beta$ and $\zeta_v = \nabla\zeta$. By inductive hypothesis, we have that $Atom(\beta)_v \vdash \nabla\beta$ and $Atom(\zeta)_v \vdash \nabla\zeta$. Note that $v(\varphi) = v(\beta \to \zeta) = 2$, so $\varphi_v = \neg\neg\varphi$. But $\varphi = \beta \to \zeta$, hence $\varphi_v = \neg\neg(\beta \to \zeta)$. We need to prove $Atom(\varphi)_v \vdash \neg\neg(\beta \to \zeta)$. By Lemma 4, we know that $\vdash (\nabla\beta \wedge \nabla\zeta) \to \neg\neg(\beta \to \zeta)$ and by inductive hypothesis, monotonicity and Rules-AND we have that: $Atom(\varphi)_v \vdash \nabla\beta \wedge \nabla\zeta$. Applying **MP** to previous statements, we conclude $Atom(\varphi)_v \vdash \neg\neg(\beta \to \zeta)$.

5. If $v(\beta) = 2$ and $v(\zeta) = 1$, then $\beta_v = \neg\neg\beta$ and $\zeta_v = \nabla\zeta$. By inductive hypothesis, we have that $Atom(\beta)_v \vdash \neg\neg\beta$ and $Atom(\zeta)_v \vdash$

$\nabla\zeta$. Note that $v(\varphi) = v(\beta \rightarrow \zeta) = 1$, so $\varphi_v = \nabla\varphi$. But $\varphi = \beta \rightarrow \zeta$, then $\varphi_v = \nabla(\beta \rightarrow \zeta)$. We need to prove $Atom(\varphi)_v \vdash \nabla(\beta \rightarrow \zeta)$. By Lemma 4, we know that $\vdash (\neg\neg\beta \wedge \nabla\zeta) \rightarrow \nabla(\beta \rightarrow \zeta)$ and by inductive hypothesis, monotonicity and Rules-AND: $Atom(\varphi)_v \vdash \neg\neg\beta \wedge \nabla\zeta$. With the application of **MP** to previous statements, we conclude $Atom(\varphi)_v \vdash \nabla(\beta \rightarrow \zeta)$.

6. If $v(\beta) = 2$ and $v(\zeta) = 0$, then $\beta_v = \neg\neg\beta$ and $\zeta_v = \sim\zeta$. By inductive hypothesis, we have that $Atom(\beta)_v \vdash \neg\neg\beta$ and $Atom(\zeta)_v \vdash \sim\zeta$. Note that $v(\varphi) = v(\beta \rightarrow \zeta) = 0$, so $\varphi_v = \sim\varphi$. But $\varphi = \beta \rightarrow \zeta$, hence $\varphi_v = \sim(\beta \rightarrow \zeta)$. We need to prove $Atom(\varphi)_v \vdash \sim(\beta \rightarrow \zeta)$. By Lemma 4, we know that $\vdash (\neg\neg\beta \wedge \sim\zeta) \rightarrow \sim(\beta \rightarrow \zeta)$ and by inductive hypothesis, monotonicity and Rules-AND we have that: $Atom(\varphi)_v \vdash \neg\neg\beta \wedge \sim\zeta$. Applying **MP** to previous statements, we conclude $Atom(\varphi)_v \vdash \sim(\beta \rightarrow \zeta)$.

□

The following lemma compiles some relevant results related to connectives $\neg$ and $\sim$.

Lemma 6. *The following formulas are theorems in* $\mathbb{L}$*:*

(a) $\vdash \sim\varphi \rightarrow (\neg\neg\varphi \rightarrow \neg\neg\neg\neg\varphi)$

(b) $\vdash \neg\neg\sim\varphi \rightarrow \neg\neg(\varphi \rightarrow \neg\neg\varphi)$

(c) $\vdash \neg\neg\varphi \rightarrow (\neg\neg\varphi \rightarrow \neg\neg\neg\neg\varphi)$

(d) $\vdash \neg\neg\varphi \rightarrow \neg\neg(\varphi \rightarrow \neg\neg\varphi)$

(e) $\vdash \neg(\varphi \rightarrow \neg\neg\varphi) \rightarrow \neg\sim\varphi$

(f) $\vdash \neg(\varphi \rightarrow \neg\neg\varphi) \rightarrow \sim\sim\varphi$

(g) $\vdash \neg(\varphi \rightarrow \neg\neg\varphi) \rightarrow \neg\varphi$

(h) $\vdash \left(\neg\varphi \wedge (\varphi \rightarrow \neg\neg\varphi) \right) \rightarrow \sim\varphi$

Only one more lemma is needed, to give the completeness proof, this lemma allows to eliminate hypotheses once it is shown that they are independent of the derivation.

Lemma 7. *Let* φ, ψ *be formulas and* Γ *be a set of formulas. If* $\Gamma, \neg\neg\varphi \vdash \psi$*;* $\Gamma, \nabla\varphi \vdash \psi$ *and* $\Gamma, \sim\varphi \vdash \psi$*; then* $\Gamma \vdash \psi$*.*

Proof. Applying Deduction theorem to $\vdash (\varphi \rightarrow \neg\neg\varphi) \rightarrow (\varphi \rightarrow \neg\neg\varphi)$, we have that, $\varphi \rightarrow \neg\neg\varphi, \varphi \vdash \neg\neg\varphi$. Through Cut to this latest formula and hypothesis $\Gamma, \neg\neg\varphi \vdash \psi$, we obtain $\Gamma, (\varphi \rightarrow \neg\neg\varphi), \varphi \vdash \psi$. On the other hand, by item h) of Lemma 6, we have, $\vdash (\neg\varphi \wedge (\varphi \rightarrow \neg\neg\varphi)) \rightarrow \sim\varphi$, now applying Lemma 1, we derive $\vdash (\varphi \rightarrow \neg\neg\varphi) \rightarrow (\neg\varphi \rightarrow \sim\varphi)$, and by Deduction theorem, we obtain $(\varphi \rightarrow \neg\neg\varphi), \neg\varphi \vdash \sim\varphi$, because of this formula and the hypothesis $\Gamma, \sim\varphi \vdash \psi$ using Cut, we conclude $\Gamma, (\varphi \rightarrow \neg\neg\varphi), \neg\varphi \vdash \psi$. At this time, we have shown: $\Gamma, (\varphi \rightarrow \neg\neg\varphi), \varphi \vdash \psi$ and $\Gamma, (\varphi \rightarrow \neg\neg\varphi), \neg\varphi \vdash \psi$; then applying Proof-by-cases, it is derived $\Gamma, (\varphi \rightarrow \neg\neg\varphi) \vdash \psi$.

On the other hand, applying Rules-AND to items f) and g) of the Lemma 6, we obtain $\neg(\varphi \rightarrow \neg\neg\varphi) \vdash (\sim\sim\varphi \wedge \neg\varphi)$, equivalently, due to the abbreviation of the connective ∇, we get $\neg(\varphi \rightarrow \neg\neg\varphi) \vdash \nabla\varphi$, then applying Cut to the last formula and the hypothesis $\Gamma, \nabla\varphi \vdash \psi$, it is concluded that $\Gamma, \neg(\varphi \rightarrow \neg\neg\varphi) \vdash \psi$.

Therefore, applying Proof-by-cases to $\Gamma, (\varphi \rightarrow \neg\neg\varphi) \vdash \psi$ and $\Gamma, \neg(\varphi \rightarrow \neg\neg\varphi) \vdash \psi$ we conclude that $\Gamma \vdash \psi$. □

Finally, we have one of the main results of this section. The proof is a consequence of Lemma 5, Lemma 7, and Theorem 1.

Theorem 4 (**Completeness of** $\mathbb{L}$). *Let* φ *be a formula. If* φ *is a tautology in* $\mathbf{CG}'_3$*, then* φ *is a theorem in* $\mathbb{L}$*.*

Proof. Suppose that φ is a tautology whose set of atomic formulas is Φ. Of the Lemma 5, we have that $\Phi_v \vdash \varphi_v$ for every 3-valuation v. Then we have two cases.

1. If $v(\varphi) = 2$, then $\Phi_v \vdash \neg\neg\varphi$, by the formula **Cw2** and **MP**, we have that $\Phi_v \vdash \varphi$. Let p any atomic formula in Φ and let $\Gamma := \Phi \setminus \{p\}$ then, we have that; $\Gamma_v, p_v \vdash \varphi$ for every v valuation. So, we obtain $\Gamma_v, \neg\neg p \vdash \varphi$; $\Gamma_v, \nabla p \vdash \varphi$, and $\Gamma_v, \sim p \vdash \varphi$. By Lemma 7, we obtain $\Gamma_v \vdash \varphi$. After $|\Phi|$ steps, we get that $\vdash \varphi$.

Table 2. Truth functions for the connectives $\vee$, $\wedge$, $\rightarrow$, and $\neg$ in Ł3

$\vee_L$	**0**	**1**	**2**
0	0	1	2
1	1	1	2
2	2	2	2

$\wedge_L$	**0**	**1**	**2**
0	0	0	0
1	0	1	1
2	0	1	2

$\rightarrow_L$	**0**	**1**	**2**
0	2	2	2
1	1	2	2
2	0	1	2

$\neg_L$	
0	2
1	1
2	0

2. If $v(\varphi) = 1$, then $\Phi_v \vdash \nabla\varphi$, by the formula $\mathbf{CG}'_3$ and **MP**, we have that $\Phi_v \vdash \varphi$. Let p any atomic formula in Φ and let $\Gamma := \Phi \setminus \{p\}$ then, we have that; $\Gamma_v, p_v \vdash \varphi$ for every valuation v. So, we have that $\Gamma_v, \neg\neg p \vdash \varphi$; $\Gamma_v, \nabla p \vdash \varphi$, and $\Gamma_v, \sim p \vdash \varphi$. By Lemma 7, we obtain $\Gamma_v \vdash \varphi$. After $|\Phi|$ steps, we get that $\vdash \varphi$.

$\square$

4 Semantical Similarities Between $\mathbf{CG}'_3$ and Ł3 Logic

Let us see now, that the logic of three values Ł3 of Łukasiewicz, and $\mathbf{CG}'_3$ has the same expressive power. To build the 3-valued logic Ł3 of Łukasiewicz, consider a propositional language $\mathcal{L}' = \{\rightarrow_L, \wedge_L, \vee_L, \neg_L, \Diamond_L, \square_L, \perp_L\}$.

Lemma 8. *In Ł3, if we consider $\rightarrow_L$ and $\perp_L$ as primitive connective we can obtain the rest of connectives as abbreviations as follows:*

$$\begin{aligned}
\varphi \vee_L \psi &:= (\varphi \rightarrow_L \psi) \rightarrow_L \psi \\
\neg_L \varphi &:= \varphi \rightarrow_L \perp_L \\
\varphi \wedge_L \psi &:= \neg_L(\neg_L \varphi \vee_L \neg_L \psi) \\
\Diamond_L \varphi &:= \neg_L \varphi \rightarrow_L \varphi \\
\square_L \varphi &:= \neg_L(\varphi \rightarrow_L \neg_L \varphi)
\end{aligned}$$

Proof. The proof follows directly from the truth tables of Ł3, see Table 2. $\square$

Lemma 9. *The connectives of $\mathbf{CG}'_3$ are definable in the connective language of Ł3.*

Proof. It is enough to show that $\neg_{\mathbf{CG}'_3}$ and $\rightarrow_{\mathbf{CG}'_3}$ are definable in terms of the connective in Ł3 since the rest of connectives have the same truth tables in both logics. Observe the following:

$$\begin{aligned}
\neg_{\mathbf{CG}'_3}\varphi &:= \neg_L \square_L \varphi. \\
\varphi \rightarrow_{\mathbf{CG}'_3} \psi &:=
\end{aligned}$$

$$(\varphi \rightarrow_L \psi) \wedge_L \square_L \neg_L(\square_L \neg_L(\square_L \neg_L(\square_L \neg_L \varphi) \rightarrow_L \psi)).$$

$\square$

Lemma 10. *The negation and implication of Ł3 are definable in terms of the connectives of $\mathbf{CG}'_3$.*

$\neg_L \varphi := (\varphi \rightarrow_{\mathbf{CG}'_3} (\varphi \wedge_{\mathbf{CG}'_3} \neg_{\mathbf{CG}'_3}\varphi)) \wedge_{\mathbf{CG}'_3} (\varphi \vee_{\mathbf{CG}'_3} (\varphi \rightarrow_{\mathbf{CG}'_3} \neg_{\mathbf{CG}'_3}\neg_{\mathbf{CG}'_3}\varphi)).$
$\varphi \rightarrow_L \psi := (\varphi \wedge_{\mathbf{CG}'_3} \neg_{\mathbf{CG}'_3}\varphi) \vee_{\mathbf{CG}'_3} (\varphi \rightarrow_{\mathbf{CG}'_3} \psi).$
Note that $\perp_L := \neg_L(\varphi \rightarrow_L \varphi).$

Theorem 5. *The connective of Ł3, are represented in terms of the connectives of $\mathbf{CG}'_3$, and vice versa.*

Proof. Direct from Lemmas 8, 9, and 10. $\square$

5 Kripke-Type Semantics for $\mathbf{CG}'_3$

In [8], Osorio et al. proved that the logic $\mathbf{G}'_3$ is an extension of the logic $\mathbf{daC}$, so it is natural to consider that Kripke models for $\mathbf{G}'_3$ are a sub collection of the Kripke models for $\mathbf{daC}$. On the other hand, for the case of $\mathbf{G}_3$, the Kripke models are Kripke models for intuitionistic but only those whose cardinality is two and the relation is a linear order, a combination of both ideas give us a characterization for $\mathbf{G}'_3$.

Definition 3. *A Kripke model for $\mathbf{G}'_3$ is a structure $\langle W, R, v \rangle$, where:*

1. *W is a set of cardinality two,*

2. *R is a linear order relation on W,*

3. *v is a valuation function of $Atom(\mathcal{L})$ to $\mathcal{P}(W)$. Given a valuation and a point w in W, we define the function $v_w : Atom(\mathcal{L}) \rightarrow \{0,1\}$ as:*

$$v_w(p) = \begin{cases} 1 & \text{if } w \in v(p), \\ 0 & \text{otherwise.} \end{cases}$$

The valuation must satisfy the following restriction for each atom p: If wRw' and $v_w(p) = 1$, then $v_{w'}(p) = 1$.
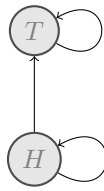
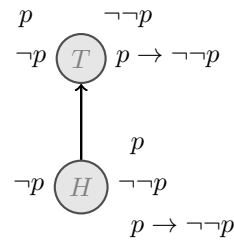**Fig. 2.** Kripke model for $\mathbf{G}'_3$

Fig. 3. Kripke countermodel for Example 1

The latter restriction imposed on valuations is called a Hereditary Property, Heredity Constraint, or Monotonicity. As we can see in [4, Proposition 2.1], the hereditary property extends to all formulas in Kripke models for $\mathbf{G}'_3$.

In analogy with the logic $\mathbf{G}_3$, we can refer to the worlds in a Kripke model for $\mathbf{G}'_3$, respectively, as H (Here) and T (There). A Kripke model for $\mathbf{G}'_3$ is a structure like the one shown in Figure 2.

Definition 4. *Let* $\mathcal{M} = \langle W, R, v \rangle$ *be a Kripke model for* $\mathbf{G}'_3$*,* $w \in W$ *and* φ *a formula.*

1. *If* $\varphi := p$ *is an atom, we have that:* $(\mathcal{M}, w) \models_{\mathbf{G}'_3} p$ *iff* $w \in v(p)$*.*

2. *If* φ *is not an atom the modeling relation is defined recursively as:*
   Let φ, ψ *be formulas and for all worlds* $w \in W$*:*

   (a) $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \varphi \wedge \psi$ *iff* $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \varphi$ *and* $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \psi$*,*

   (b) $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \varphi \vee \psi$ *iff* $(M, w) \models_{\mathbf{G}'_3} \varphi$ *or* $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \psi$*,*

   (c) $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \varphi \rightarrow \psi$ *iff for all* w' *such that* wRw'*, if* $(\mathcal{M}, w') \models_{\mathbf{G}'_3} \varphi$ *then* $(\mathcal{M}, w') \models_{\mathbf{G}'_3} \psi$*,*

   (d) $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \neg\varphi$ *iff there exists* w' *such that* $w'Rw$*,* $(\mathcal{M}, w') \not\models_{\mathbf{G}'_3} \varphi$*.*

We say a formula φ is valid in $\mathcal{M}$ and we write $\mathcal{M} \models_{\mathbf{G}'_3} \varphi$, if and only if, for every $w \in W$, $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \varphi$.

Example 1. *Logic* $\mathbf{G}'_3$ *does not validate the formula* $p \rightarrow \neg\neg p$ *but it validates the formula* $\neg\neg p \rightarrow p$*.*

Proof. Suppose $W = \{H, T\}$ is the set of worlds, the relation is $R = \{\langle H, H \rangle, \langle H, T \rangle, \langle T, T \rangle\}$ and lets $v_T(p) = 1$, and for any other variable and point the valuation is 0, the model is depicted, in the Figure 3. The formula $p \rightarrow \neg\neg p$ is not valid at H and T in the model. On the other hand, the formula $\neg\neg p \rightarrow p$ is valid in all Kripke model for $\mathbf{G}'_3$. Indeed, suppose otherwise. Then there is a model such that $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \neg\neg p$ and $(\mathcal{M}, w) \not\models_{\mathbf{G}'_3} p$ for some $w \in W$. We know that $(\mathcal{M}, w) \models_{\mathbf{G}'_3} \neg\neg p$, so there is $w' \in W$ for which $w'Rw$ and $(\mathcal{M}, w') \not\models_{\mathbf{G}'_3} \neg p$ and this is, for all $w'' \in W$ for which $w''Rw'$ and $(\mathcal{M}, w'') \models_{\mathbf{G}'_3} p$. By the definition of valuation, we must have $(\mathcal{M}, w) \models_{\mathbf{G}'_3} p$, which is a contradiction. $\square$

Given the narrow relation between $\mathbf{G}'_3$ and $\mathbf{CG}'_3$, it is natural to think that if there is a Kripke-type semantics for the latter, its semantic must be closely related to that of $\mathbf{G}'_3$.

We can define a type for Kripke semantics to $\mathbf{CG}'_3$ in two different ways. The first based on the semantics of $\mathbf{G}'_3$, and the second redefining the notion of validity as discussed below.

5.1 Semantics-based on $\mathbf{G}'_3$ Semantics

Definition 5. *Let* $\mathcal{M} = \langle W, R, v \rangle$ *be a Kripke model for* $\mathbf{G}'_3$*,* $w \in W$ *and* φ *a formula. We define the modeling relation (denoted by* $\models_{\mathbf{CG}'_3}$*) as follows:*

$$(\mathcal{M}, w) \models_{\mathbf{CG}'_3} \varphi \text{ if and only if there is } wRw' \text{ such that } (\mathcal{M}, w') \models_{\mathbf{G}'_3} \varphi.$$

As we can see, the hereditary property also holds for $\models_{\mathbf{CG}'_3}$.

Theorem 6. *If* $(\mathcal{M}, x) \models_{\mathbf{CG}'_3} \varphi$ *and* xRy*, then* $(\mathcal{M}, y) \models_{\mathbf{CG}'_3} \varphi$*.*

Proof. The proof is by induction on the length of the formula φ. □

The following theorem establishes an equivalence between many-valued semantics and Kripke semantics for $\mathbf{CG}'_3$.

Proposition 2. *Let φ be a formula on the language of $\mathbf{CG}'_3$. There exists an interpretation $t : \mathcal{L} \to \{0, 1, 2\}$ such that $t(\varphi) = 0$, if and only if there is a Kripke model for $\mathbf{CG}'_3$ whose valuation v is such that $v(\varphi) = \emptyset$.*

Proof. The proof is by induction on the length of the formula φ. □

Theorem 7. *Let φ be a formula in the language of $\mathbf{CG}'_3$, then:*

$$\models_{\mathbf{CG}'_3} \varphi \text{ if and only if for any Kripke model } \mathcal{M} \text{ for }$$
$$\mathbf{CG}'_3\text{, it holds that } \mathcal{M} \models_{\mathbf{CG}'_3} \varphi.$$

Proof. The proof is by induction on the length of the formula φ and applying Proposition 2. □

5.2 Semantics Redefining the Validity Concept

An alternative way of defining the modeling relation for $\mathbf{CG}'_3$ is to consider that the models for $\mathbf{CG}'_3$ are those for $\mathbf{G}'_3$ but changing the Modeling Definition. In [2], the authors explain the notion of being e-valid to the characterization of the validity depends on an existential connective and to distinguish the concept of validity.

Definition 6. *A formula φ is said to be **e-valid** on a model $\mathcal{M}$ for logic $\mathbf{CG}'_3$ if exists a point x in $\mathcal{M}$ such that $(\mathcal{M}, x) \models_{\mathbf{G}'_3} \varphi$.*

It is easy to check that this new definition changing the notion of validity coincides with the preceding one.

Lemma 11. *Let φ be a formula in the language of $\mathbf{CG}'_3$, then:*

$$\models_{\mathbf{CG}'_3} \varphi \text{ if and only if for any Kripke model } \mathcal{M} \text{ for }$$
$$\mathbf{CG}'_3\text{, it holds that } \varphi \text{ is e-valid.}$$

6 Conclusion and Future Work

Logic $\mathbf{CG}'_3$ was defined in [13] utilizing semantics. In this paper, the authors present the logic $\mathbf{CG}'_3$ from a semantic, and syntactic point of view, his contributions are summarized as follows:

1. A Hilbert type axiomatization for $\mathbf{CG}'_3$ using the Kalmar technique, this axiomatic system satisfies many properties, such as those presented in Theorem 1 and Lemma 1. Among these properties, we can find Deduction theorem, Cut, Rules-AND, among other things. Through this axiomatization, we show that Ł3 and $\mathbf{CG}'_3$ have the same expressive power, Theorem 5.

2. A characterization of $\mathbf{CG}'_3$ using Kripke models. Thanks to the Kripke semantics for these logics, they obtained a new tool that can help us have a better understanding of paraconsistent logics.

There are some relevant issues associated with the $\mathbf{CG}'_3$ system that needs to be studied. For example, in the semantics approach, there is a many-valued characterization for $\mathbf{CG}'_3$, but an algebraic approach to $\mathbf{CG}'_3$ is still missing. In [3] and [5], we can find some algebraic methods such as Blok-Pigozzi and Fidel structures, respectively, that can help the study of these semantics applied to the logic.

Acknowledgments

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